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The center of the affine nilTemperley–Lieb algebra

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Abstract We give a description of the center of the affine nilTemperley–Lieb algebra based on a certain grading of the algebra and on a faithful representation of it on fermionic particle configurations. We present a normal form for monomials, hence construct a basis of the algebra, and use this basis to show that the affine nilTemperley–Lieb algebra is finitely generated over its center. As an application, we obtain a natural embedding of the affine nilTemperley–Lieb algebra on N generators into the affine nilTemperley–Lieb algebra on $N+1$ generators.

1 Introduction

The main goal of this work is to describe the center of the affine nilTemperley–Lieb algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$ over any ground field. Only two tools are used: a fine grading on $\mathrm{n}\widehat{\mathrm{TL}}_N$ and a representation of $\mathrm{n}\widehat{\mathrm{TL}}_N$ on fermionic particle configurations on a circle. It is essential that this graphical representation be faithful (see [12, Prop. 9.1]). We provide an alternative proof of that fact by constructing a basis for $\mathrm{n}\widehat{\mathrm{TL}}_N$ that is especially adapted to the problem. This basis has further advantages: It can be used to prove that the affine nilTemperley–Lieb algebra is

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finitely generated over its center. Also, it can be used to exhibit an explicit embedding of $\mathrm{n}\widehat{\mathrm{TL}}_N$ into $\mathrm{n}\widehat{\mathrm{TL}}_{N+1}$ defined on basis elements that otherwise would not be apparent, since the defining relations of these algebras are affine, and there is no embedding of the corresponding Coxeter graphs.

For a ground field $\mathbb{k}$, the *affine nilTemperley–Lieb algebra* $\mathrm{n}\widehat{\mathrm{TL}}_N$ is the unital associative $\mathbb{k}$-algebra given by N generators $a_0, \dots, a_{N-1}$ and nil relations $a_i^2 = 0$ and $a_i a_{i\pm1} a_i = 0$ for all i. Generators that are far apart commute, i.e. $a_i a_j = a_j a_i$ for $i - j \not\equiv \pm 1 \bmod N$. In these relations, the indices are interpreted modulo N so that the generators a_0 and a_{N-1} are neighbours that do not commute. The subalgebra of $\mathrm{n}\widehat{\mathrm{TL}}_N$ generated by $a_1, \dots, a_{N-1}$ is the *(finite) nilTemperley–Lieb algebra* nTL_N, as in [19]. The affine nilTemperley–Lieb algebra appears in many different settings, which we describe next.

1. $\mathrm{n}\widehat{\mathrm{TL}}_N$ *is a quotient of the affine nilCoxeter algebra of type* $\tilde{\mathsf{A}}_{N-1}$.
   The affine nilCoxeter algebra $\widehat{\mathsf{U}}_N$ of type $\tilde{\mathsf{A}}_{N-1}$ over a field $\mathbb{k}$ is the unital associative algebra generated by elements u_i, $0 \le i \le N-1$, satisfying the relations $u_i^2 = 0$; $u_i u_j = u_j u_i$ for $i - j \not\equiv \pm 1 \bmod N$; and $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ for $1 \le i \le N-1$, where the subscripts are read modulo N. The algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$ is isomorphic to the quotient of $\widehat{\mathsf{U}}_N$ obtained by imposing the additional relations $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} = 0$ for $1 \le i \le N-1$. The affine nilCoxeter algebra is closely connected with affine Schur functions, k-Schur functions, and the affine Stanley symmetric functions, which are related to reduced word decompositions in the affine symmetric group (see e.g. [14,15]). The nilCoxeter algebra U_N has generators u_i, $1 \le i \le N-1$, which satisfy the same relations as they do in $\widehat{\mathsf{U}}_N$. It first appeared in work on the cohomology of flag varieties [3] and has played an essential role in studies on Schubert polynomials, Stanley symmetric functions, and the geometry of flag varieties (see for example [8,11,16,17]). The definition of U_N was inspired by the divided difference operators ∂_i on polynomials in variables $\mathbf{x} = \{x_1, \dots, x_N\}$ defined by

   $$\partial_i(f) = \frac{f(\mathbf{x}) - f(\sigma_i \mathbf{x})}{x_i - x_{i+1}},$$

   where the transposition σ_i fixes all the variables except for x_i and x_{i+1}, which it interchanges. The operators ∂_i satisfy the nilCoxeter relations above, and applications of these relations enabled Fomin and Stanley [8] to recover known properties and establish new properties of Schubert polynomials.
   The algebra U_N belongs to a two-parameter family of algebras having generators u_i, $1 \le i \le N-1$, which satisfy the relations $u_i u_j = u_j u_i$ for $|i-j| > 1$ and $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ for $1 \le i \le N-2$ from above, together with the relation $u_i^2 = \alpha u_i + \beta$ for all i, where α, β are fixed parameters. In particular, the specialization $\alpha = \beta = 0$ yields the nilCoxeter algebra; $\alpha = 0$, $\beta = 1$ gives the standard presentation of the group algebra of the symmetric group $\mathbb{k}\mathsf{S}_N$; and $\alpha = q - 1$, $\beta = q$ gives the Hecke algebra $\mathsf{H}_N(q)$ of type A.
   Motivated by categorification results in [6], Khovanov [10] introduced restriction and induction functors F_D and F_X corresponding to the natural inclusion of algebras $\mathsf{U}_N \hookrightarrow \mathsf{U}_{N+1}$ on the direct sum $\mathcal{C}$ of the categories $\mathcal{C}_N$ of finite-dimensional U_N-modules. These functors categorify the Weyl algebra of differential operators with polynomial coefficients in one variable and correspond to the Weyl algebra generators ∂ and x (derivative and multiplication by x), which satisfy the relation $\partial x - x\partial = 1$.
   Brichard [5] used a diagram calculus on cylinders to determine the dimension of the center of U_N and to describe a basis of the center for which the multiplication is trivial.

In this diagram calculus on N strands, the generator u_i corresponds to a crossing of the strands i and $i+1$. The nil relation $u_i^2 = 0$ is represented by demanding that any two strands may cross at most once; otherwise the diagram is identified with zero.

2. $\mathrm{n}\widehat{\mathrm{TL}}_N$ *is a quotient of the negative part of the universal enveloping algebra of the affine Lie algebra* $\widehat{\mathfrak{sl}}_N$.
The negative part U^- of the universal enveloping algebra U of the affine Lie algebra $\widehat{\mathfrak{sl}}_N$ has generators f_i, $0 \le i \le N-1$, which satisfy the Serre relations

$$\begin{aligned} &f_i^2 f_{i+1} - 2 f_i f_{i+1} f_i + f_{i+1} f_i^2 = 0 \\ &\quad = f_{i+1}^2 f_i - 2 f_{i+1} f_i f_{i+1} + f_i f_{i+1}^2 \ \text{ and } \ f_i f_j = f_j f_i \text{ for } i-j \neq \pm 1 \bmod N \end{aligned}$$

(all indices modulo N). Factoring U^- by the ideal generated by the elements f_i^2, $0 \le i \le N-1$, gives $\mathrm{n}\widehat{\mathrm{TL}}_N$ whenever the characteristic of $\Bbbk$ is different from 2.

3. $\mathrm{n}\widehat{\mathrm{TL}}_N$ *acts on the small quantum cohomology ring of the Grassmannian.*
As in [19, Sec. 2], (see also [12]), consider the cohomology ring $\mathrm{H}^\bullet(\mathrm{Gr}(k, N))$ with integer coefficients for the Grassmannian $\mathrm{Gr}(k, N)$ of k-dimensional subspaces of $\Bbbk^N$. It has a basis given by the Schubert classes $[\Omega_\lambda]$, where λ runs over all partitions with k parts, the largest part having size $N-k$. By recording the k vertical and $N-k$ horizontal steps that identify the Young diagram of λ inside the northwest corner of a $k \times (N-k)$ rectangle, such a partition corresponds to a $(0,1)$-sequence of length N with k ones (resp. $N-k$ zeros) in the positions corresponding to the vertical (respectively horizontal) steps. As a $\mathbb{Z}[q]$-module for an indeterminate q, the quantum cohomology ring of the Grassmannian is given by $\mathrm{qH}^\bullet(\mathrm{Gr}(k, N)) = \mathbb{Z}[q] \otimes_{\mathbb{Z}} \mathrm{H}^\bullet(\mathrm{Gr}(k, N))$ together with a q-multiplication. The $\mathrm{n}\widehat{\mathrm{TL}}_N$-action can be defined combinatorially on

$$\mathrm{qH}^\bullet(\mathrm{Gr}(k, N)) \ \cong \ \mathrm{span}_{\mathbb{Z}[q]} \{(0,1)\text{-sequences of length } N \text{ with } k \text{ ones}\}$$

as described in the next item, and the multiplication of two Schubert classes $[\Omega_\lambda] \cdot [\Omega_\mu]$ is equal to $s_\lambda \cdot [\Omega_\mu]$ where s_λ is a certain Schur polynomial in the generators of $\mathrm{n}\widehat{\mathrm{TL}}_N$ as in [19, Cor. 8.3].

4. $\mathrm{n}\widehat{\mathrm{TL}}_N$ *acts faithfully on fermionic particle configurations on a circle.*
This is the graphical representation from [12] (see also [19]), which we use in our description of the center of $\mathrm{n}\widehat{\mathrm{TL}}_N$. First, a $(0,1)$-sequence with k ones is identified with a circular particle configuration having N positions, where the k particles are distributed at the position on the circle that corresponds to their position in the sequence, so that there is at most one particle at each position. On the space

$$\mathrm{span}_{\Bbbk[q]} \{\text{fermionic particle configurations of } k \text{ particles on a circle with } N \text{ positions}\},$$

the generators a_i of $\mathrm{n}\widehat{\mathrm{TL}}_N$ act by sending a particle lying at position i to position $i+1$. Additionally, the particle configuration is multiplied by $\pm q$ when applying a_0. The precise definition is given in Sect. 4, but here is a representative picture (Fig. 1).

5. $\mathrm{n}\widehat{\mathrm{TL}}_N$ *appears as a subalgebra of the annihilation/creation algebra.*
The finite nilTemperley–Lieb algebra is a subalgebra of the Clifford algebra having generators $\{\xi_i, \xi_i^* \mid 0 \le i \le N-1\}$ and relations $\xi_i \xi_j + \xi_j \xi_i = 0$, $\xi_i^* \xi_j^* + \xi_j^* \xi_i^* = 0$, $\xi_i \xi_j^* + \xi_j^* \xi_i = \delta_{ij}$. The Clifford generators ξ_i (resp. ξ_i^*) act on the fermionic particle configurations by annihilation (resp. creation) of a particle at position i. The finite nilTemperley–Lieb algebra appears inside the Clifford algebra via $a_i \mapsto \xi_{i+1}^* \xi_i$. As discussed in [12, Sec. 8], the affine nilTemperley–Lieb algebra is a q-deformation of this construction.

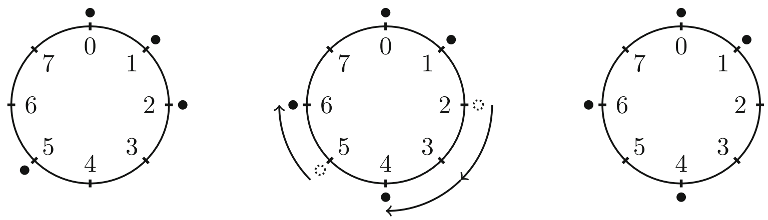

Fig. 1 $N = 8$: Application of $a_3a_2a_5$ to the particle configuration $(0, 1, 2, 5)$ gives $(0, 1, 4, 6)$

6. $\mathrm{n}\widehat{\mathrm{TL}}_N$ *is the associated graded algebra of the affine Temperley–Lieb algebra.*
The affine Temperley–Lieb algebra $\widehat{\mathrm{TL}}_N(\delta)$ has the usual commuting relations and the relations $a_ia_{i\pm1}a_i = a_i$ and $a_i^2 = \delta a_i$ for some parameter $\delta \in \Bbbk$ instead of the nil relations (where again all indices are mod N). It is a filtered algebra with its ℓth filtration space generated by all monomials of length $\leq \ell$. Since its associated graded algebra is $\mathrm{n}\widehat{\mathrm{TL}}_N$ for any value of δ, elements of $\mathrm{n}\widehat{\mathrm{TL}}_N$ can be identified with reduced expressions in $\widehat{\mathrm{TL}}_N(\delta)$.
The diagrammatic structure of $\widehat{\mathrm{TL}}_N(\delta)$ is given by the same pictures as for the Temperley–Lieb algebra, but now the diagrams are wrapped around the cylinder (see e.g. [7,13]). The top and bottom of the cylinder each have N nodes. Monomials in the affine Temperley–Lieb algebra are represented by diagrams of N non-crossing strands, each connecting a pair of those $2N$ nodes. Multiplication of two monomials is realized by stacking the cylinders one on top of the other, and connecting and smoothing the strands. Whenever the strands form a circle, this is removed from the diagram at the expense of multiplying by the parameter δ. The relation $a_ia_{i\pm1}a_i = a_i$ corresponds to the isotopy between a strand that changes direction and a strand that is pulled straight.
In contrast, this diagrammatic realization for the affine <u>nil</u>Temperley–Lieb algebra would not respect isotopy: The relation $a_ia_{i\pm1}a_i = 0$ implies that strands which change the direction are identified with zero. Nevertheless, the diagram of a reduced expression in $\widehat{\mathrm{TL}}_N$ may be considered as an element of $\mathrm{n}\widehat{\mathrm{TL}}_N$. Such a diagram consists of a number (possibly 0) of arcs that connect two nodes on the top of the cylinder, the same number of arcs connecting two nodes on the bottom, and arcs that connect a top node and a bottom one. The latter arcs wrap around the cylinder either all in a strictly clockwise direction or all in a strictly counterclockwise way. Since the multiplication of two such diagrams may give zero, we will not use this diagrammatic realization here.

We proceed as follows: In Sect. 2, we introduce the notation used in this article. The $\mathbb{Z}^N$-grading of $\mathrm{n}\widehat{\mathrm{TL}}_N$ is given is Sect. 3, and its importance for the description of the center is discussed. In Sect. 4, we give a detailed definition of the $\mathrm{n}\widehat{\mathrm{TL}}_N$-action on particle configurations on a circle. We also define special monomials that serve as the projections onto a single particle configuration (up to multiplication by $\pm q$). Theorem 4.5 of that section recalls [12, Prop. 9.1] stating that the representation is faithful. In [12], this fact is deduced from the finite nilTemperley–Lieb algebra case, as treated in [4] and [2, Prop. 2.4.1]. We give a complete, self-contained proof in Sect. 8. Our proof is elementary and relies on the construction of a basis. Section 5 contains the main result (Theorem 5.5) of this article:

Theorem *The center of $n\widehat{TL}_N$ is the subalgebra*

$$\mathsf{C}_N = \mathsf{Cent}(n\widehat{TL}_N) = \langle 1, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_{N-1} \rangle \cong \frac{\Bbbk[\boldsymbol{t}_1, \ldots, \boldsymbol{t}_{N-1}]}{(\boldsymbol{t}_k\boldsymbol{t}_\ell \mid k \neq \ell)},$$

where the generator $\boldsymbol{t}_k = (-1)^{k-1} \sum_{|\mathbf{I}|=k} a(\hat{\mathbf{I}})$ *is the sum of monomials* $a(\hat{\mathbf{I}})$ *corresponding to particle configurations given by increasing sequences* $\mathbf{I} = \{1 \le i_1 < \cdots < i_k \le N\}$ *of length* k*. The monomial* $a(\hat{\mathbf{I}})$ *sends particle configurations with* $n \neq k$ *particles to 0 and acts on a particle configuration with* k *particles by projecting onto* $\mathbf{I}$ *and multiplying by* $(-1)^{k-1}q$*. Hence,* $\boldsymbol{t}_k$ *acts as multiplication by* q *on the configurations with* k *particles.*

Our $N-1$ central generators $\mathbf{t}_k$ are essentially the $N-1$ central elements constructed by Postnikov. Lemma 9.4 of [19] gives an alternative description of $\mathbf{t}_k$ as product of the kth elementary symmetric polynomial (with factors cyclically ordered) with the $(N-k)$th complete homogeneous symmetric polynomial (with factors reverse cyclically ordered) in the noncommuting generators of $\mathrm{n}\widehat{\mathrm{TL}}_N$. The above theorem shows that in fact these elements generate the entire center of $\mathrm{n}\widehat{\mathrm{TL}}_N$. In Sect. 6, we establish that $\mathrm{n}\widehat{\mathrm{TL}}_N$ is finitely generated over its center. In Sect. 7, we define a monomial basis for $\mathrm{n}\widehat{\mathrm{TL}}_N$ indexed by pairs of particle configurations together with a natural number indicating how often the particles have been moved around the circle. A proof that this is indeed a basis of $\mathrm{n}\widehat{\mathrm{TL}}_N$ can be found in Sect. 8. With this basis at hand, we obtain inclusions $\mathrm{n}\widehat{\mathrm{TL}}_N \subset \mathrm{n}\widehat{\mathrm{TL}}_{N+1}$. The inclusions are not as obvious as those for the nilCoxeter algebra U_N having underlying Coxeter graph of type A_{N-1}, since one cannot deduce them from embeddings of the affine Coxeter graphs. Our result, Theorem 7.1, reads as follows:

Theorem *For all* $0 \le m \le N-1$*, there are unital algebra embeddings* $\varepsilon_m : n\widehat{TL}_N \to n\widehat{TL}_{N+1}$ *given by*

$$a_i \mapsto a_i \text{ for } 0 \le i \le m-1, \qquad a_m \mapsto a_{m+1}a_m, \quad a_i \mapsto a_{i+1} \text{ for } m+1 \le i \le N-1.$$

In Sect. 8, we show how to construct the monomial basis, namely by using a normal form algorithm that reorders the factors of a nonzero monomial. Our basis is reminiscent of the Jones normal form for reduced expressions of monomials in the Temperley–Lieb algebra, as discussed in [20], and is characterised in Theorem 8.6 as follows: (See also Theorem 7.5 which gives a different description.)

Theorem (Normal form) *Every nonzero monomial in the generators* a_j *of* $n\widehat{TL}_N$ *can be rewritten uniquely in the form*

$$(a_{i_1}^{(m)} \dots a_{i_k}^{(m)}) \dots (a_{i_1}^{(n+1)} \dots a_{i_k}^{(n+1)})(a_{i_1}^{(n)} \dots a_{i_k}^{(n)}) \dots (a_{i_1}^{(1)} \dots a_{i_k}^{(1)})(a_{i_1} \dots a_{i_k})$$

with $a_{i_\ell}^{(n)} \in \{1, a_0, a_1, \dots, a_{N-1}\}$ *for all* $1 \le n \le m,\ 1 \le \ell \le k$*, such that*

$$a_{i_\ell}^{(n+1)} \in \begin{cases} \{1\} & \text{if } \ a_{i_\ell}^{(n)} = 1, \\ \{1, a_{j+1}\} & \text{if } \ a_{i_\ell}^{(n)} = a_j. \end{cases}$$

The factors $a_{i_1}, \dots, a_{i_k}$ *are determined by the property that the generator* $a_{i_\ell - 1}$ *does not appear to the right of* a_{i_ℓ} *in the original presentation of the monomial. Alternatively, every nonzero monomial is uniquely determined by the following data from its action on the graphical representation:*

- *the input particle configuration with the minimal number of particles on which it acts nontrivially,*
- *the output particle configuration,*
- *the power of* q *by which it acts.*

For the proof of this result, we recall a characterisation of the nonzero monomials in $\mathrm{n}\widehat{\mathrm{TL}}_N$ from [9]. Then we prove faithfulness of the graphical representation of $\mathrm{n}\widehat{\mathrm{TL}}_N$ by describing explicitly the matrices representing our basis elements. Al Harbat [1] has recently described a normal form for fully commutative elements of the affine Temperley–Lieb algebra, which gives a different normal form when passing to $\mathrm{n}\widehat{\mathrm{TL}}_N$.

Our results hold over an arbitrary ground field $\Bbbk$, even one of characteristic 2, simply by ignoring signs in that case. In fact, our arguments work for any associative commutative unital ground ring R by replacing $\Bbbk$-vector spaces and $\Bbbk$-algebras with free R-modules and R-algebras, respectively. In particular, the affine nilTemperley–Lieb algebra over $\Bbbk$ is replaced by the R-algebra with the same generators and relations, and the polynomial ring $\Bbbk[q]$ is replaced by $R[q]$. We can even drop the assumption that the ring R is commutative if we slightly modify the statements about the center. This is possible because our arguments mainly rely on investigating monomials in the generators of $\mathrm{n}\widehat{\mathrm{TL}}_N$. However, for simplicity we have chosen to assume $\Bbbk$ is a field throughout the article.

2 Notation

Let $\Bbbk$ be any field, and assume N is a positive integer. The *affine nilTemperley–Lieb algebra* $\mathrm{n}\widehat{\mathrm{TL}}_N$ of rank N is the unital associative $\Bbbk$-algebra generated by elements $a_0, \dots, a_{N-1}$ subject to the defining relations

$$\begin{aligned} a_i^2 &= 0 && \text{for all } 0 \le i \le N-1, \\ a_i a_j &= a_j a_i && \text{for all } i-j \ne \pm 1 \bmod N, \\ a_i a_{i+1} a_i &= a_{i+1} a_i a_{i+1} = 0 && \text{for all } 0 \le i \le N-1, \end{aligned}$$

where all indices are taken modulo N, so in particular $a_{N-1}a_0a_{N-1} = a_0a_{N-1}a_0 = 0$. The *finite nilTemperley–Lieb algebra* nTL_N, as defined in [19], is the subalgebra of $\mathrm{n}\widehat{\mathrm{TL}}_N$ generated by $a_1, \dots, a_{N-1}$ (or in fact, by any $N-1$ of the generators a_i). We adopt the convention that $\mathrm{nTL}_1 = \Bbbk 1$. We fix the following notation for monomials in $\mathrm{n}\widehat{\mathrm{TL}}_N$ and nTL_N: For an ordered index sequence $\underline{j} = (j_1, \dots, j_m)$ with $0 \le j_1, \dots, j_m \le N-1$, we define the ordered monomial $a(\underline{j}) = a_{j_1} \dots a_{j_m}$. Unless otherwise specified, we use the letters i, j for indices from $\mathbb{Z}/N\mathbb{Z}$; in particular, we often identify the indices 0 and N.

Throughout we will assume $N \ge 3$.

3 Gradings

One of the ingredients needed in Sect. 5 to study the center of $\mathrm{n}\widehat{\mathrm{TL}}_N$ is a grading on the algebra.

Gradings faciliate the computation of the center of an algebra, as the following standard result reduces the work to determining homogeneous central elements.

Lemma 3.1 *If $A = \bigoplus_{g\in G} A_g$ is an algebra graded by some abelian group G, then the center of A is homogeneous, i.e. it inherits the grading.*

Proof Let $a = \sum_{g\in G} a_g$ be a central element of the graded algebra $A = \bigoplus_{g\in G} A_g$. We have for $b_h \in A_h$ that

$$\sum_{g\in G} a_g b_h = a b_h = b_h a = \sum_{g\in G} b_h a_g.$$

Since this equality must hold in every graded component, we get $a_g b_h = b_h a_g$ for all homogeneous elements b_h. Now take any element $b = \sum_{h \in G} b_h$ in A, then

$$a_g b = \sum_{h \in G} a_g b_h = \sum_{h \in G} b_h a_g = b a_g,$$

hence a_g is central. □

Since the defining relations are homogeneous, both $\mathrm{n}\widehat{\mathrm{TL}}_N$ and nTL_N have a $\mathbb{Z}$-grading by the length of a monomial, i.e. all generators a_i have $\mathbb{Z}$-degree 1. This can be refined to a $\mathbb{Z}^N$-grading by assigning to the generator a_i the degree ζ_i, the ith standard basis vector in $\mathbb{Z}^N$. In either grading, we say that the degree 0 part of an element in $\mathrm{n}\widehat{\mathrm{TL}}_N$ or nTL_N is its constant term.

The $\mathbb{Z}^N$-grading is finer than the $\mathbb{Z}$-grading in the sense that any $\mathbb{Z}$-graded component of degree different from 0 decomposes into a sum of $\mathbb{Z}^N$-graded components of strictly smaller dimension.

Remark 3.2 Why do we exclude the case of $N \leq 2$ from our considerations? For $N = 1, 2$, there are isomorphisms $\mathrm{n}\widehat{\mathrm{TL}}_N \cong \mathrm{nTL}_{N+1}$, and in these cases the center is uninteresting. The algebra $\mathrm{n}\widehat{\mathrm{TL}}_1$ is 2-dimensional and commutative; while $\mathrm{n}\widehat{\mathrm{TL}}_2$ has dimension 5, and its center can be computed by hand making use of Lemma 3.1 and can be shown to be the $\Bbbk$-span of $1, a_0 a_1, a_1 a_0$.

Remark 3.3 The affine (or finite) Temperley–Lieb algebra, which has relations $a_i a_j = a_j a_i$ for $i - j \neq \pm 1 \pmod N$, $a_i a_{i\pm 1} a_i = a_i$, and $a_i^2 = \delta a_i$ for some $\delta \in \Bbbk$, is a filtered algebra with respect to the length filtration. For this algebra, the ℓth filtration space is generated by all monomials of length $\leq \ell$. Its associated graded algebra is $\mathrm{n}\widehat{\mathrm{TL}}_N$ (or nTL_N). Thus, $\mathrm{n}\widehat{\mathrm{TL}}_N$ is infinite dimensional when $N \geq 3$, while nTL_N has dimension equal to the Nth Catalan number $\frac{1}{N+1}\binom{2N}{N}$.

4 A faithful representation

The second ingredient we use to determine the center is a faithful representation of $\mathrm{n}\widehat{\mathrm{TL}}_N$. Here we recall the definition of the representation from [12] and describe its graphical realization, which is very convenient to work with.

Fix a basis $v_1, \ldots, v_N$ of $\Bbbk^N$. Consider the vector space $\mathsf{V} = \bigoplus_{k=0}^{N} \left(\Bbbk[q] \otimes \bigwedge^k \Bbbk^N \right)$. It has a standard $\Bbbk[q]$-basis consisting of wedges

$$v(\mathbf{I}) := v_{i_1} \wedge \cdots \wedge v_{i_k} \quad \text{for all (strictly) increasing sequences } \mathbf{I} = \{1 \leq i_1 < \cdots < i_k \leq N\}$$

for all $0 \leq k \leq N$, where the basis element of $\Bbbk = \bigwedge^0 \Bbbk^N$ is denoted $v(\emptyset)$. Throughout the rest of the paper, all tensor products are taken over $\Bbbk$, and we omit the tensor symbol in $\Bbbk[q]$-linear combinations of wedges.

Remark 4.1 The indices of the vectors v_j should be interpreted modulo N. We make no distinction between v_0 and v_N and often use the two interchangeably.

It is helpful to visualize the basis elements $v(\mathbf{I})$ as particle configurations having $0 \leq k \leq N$ particles arranged on a circle with N positions, where there is at most one particle at each site, as pictured below for $N = 8$ and $v(1, 5, 6) = v_1 \wedge v_5 \wedge v_6$ (Fig. 2). The vector $v(\emptyset)$

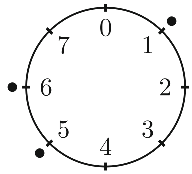

Fig. 2 The element $v_1 \wedge v_5 \wedge v_6$ in the graphical realization

corresponds to the configuration with no particles. Then V is the $\Bbbk[q]$-span of such circular particle configurations.

There is an action of the affine nilTemperley–Lieb algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$ defined on the basis vectors $v(\mathbf{I})$ of V as follows:

Definition 4.2 For $1 \leq j \leq N-1$,

$$a_j v(\mathbf{I}) = \begin{cases} v_{i_1} \wedge \cdots \wedge v_{i_{\ell-1}} \wedge v_{j+1} \wedge v_{i_{\ell+1}} \wedge \cdots \wedge v_{i_k}, & \text{if } i_\ell = j \text{ for some } \ell, \\ 0, & \text{otherwise.} \end{cases}$$

For the action of a_0, note that v_N appears in the basis element $v(\mathbf{I})$ if and only if it occurs in the last position, i.e. $v_{i_k} = v_N$, and define

$$a_0 v(\mathbf{I}) \;=\; \begin{cases} (-1)^{k-1} q \cdot v_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}, & \text{if } i_k = N, \\ 0, & \text{otherwise.} \end{cases}$$

The sign appears in $a_0 v(\mathbf{I})$ because of the equality

$$q \cdot v_{i_1} \wedge \cdots \wedge v_{i_{k-1}} \wedge v_1 \;=\; (-1)^{k-1} q \cdot v_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}.$$

Remark 4.3 It follows that $a_j v(\mathbf{I}) = 0$ if the sequence $\mathbf{I}$ contains $j+1$ or if it does not contain j. In other words, a_j acts by replacing v_j by v_{j+1}. If this creates a wedge expression with two factors equal to v_{j+1}, the result is zero. Thus, for any monomial $a(\underline{j})$ there is a unique increasing sequence $\mathbf{J} = \{1 \leq j_1 < \cdots < j_k \leq N\}$ with k minimal on which the monomial acts nontrivially.

In the graphical description, a_j moves a particle clockwise from position j to position $j+1$, and one records 'passing position 0' by multiplying by $\pm q$ as illustrated by the particle configurations in Fig. 3.

It is easy to verify that the defining relations for $\mathrm{n}\widehat{\mathrm{TL}}_N$ hold for this action, assuming that $N \geq 3$. Hence we obtain

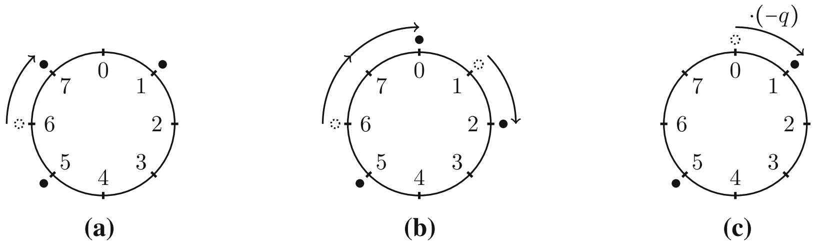

Fig. 3 Examples for the action of $\mathrm{n}\widehat{\mathrm{TL}}_N$ on a particle configuration. **a** $a_6(v_1 \wedge v_5 \wedge v_6) = v_1 \wedge v_5 \wedge v_7$, **b** $a_7 a_1 a_6(v_1 \wedge v_5 \wedge v_6) = v_2 \wedge v_5 \wedge v_0$, **c** $a_0(v_5 \wedge v_0) = -q \cdot v_1 \wedge v_5$

Lemma 4.4 (a) *Definition 4.2 gives a representation of* $n\widehat{TL}_N$ *on* V.

(b) *The number of wedges (i.e., the number of particles) remains constant under the action of the generators* a_i, *so that* $\mathsf{V} = \bigoplus_{k=0}^{N} \left(\Bbbk[q] \otimes \bigwedge^k \Bbbk^N\right)$ *is a direct sum decomposition of* V *as an* $n\widehat{TL}_N$*-module.*

The following crucial statement is taken from [2, Prop. 2.4.1] and [12, Prop. 9.1.(2)]. We will give a detailed proof adapted to our notation in Sect. 8.

Theorem 4.5 *The action from Definition 4.2 gives a faithful representation of* $n\widehat{TL}_N$ *on* V *when* $N \geq 3$.

From now on, we will identify elements of $\mathrm{n}\widehat{\mathrm{TL}}_N$ with their action on the particle configurations of the graphical representation.

Remark 4.6 The spaces $\Bbbk[q] \otimes \bigwedge^0 \Bbbk^N$ and $\Bbbk[q] \otimes \bigwedge^N \Bbbk^N$ are trivial summands in V on which every generator a_i acts as 0, and so they may be ignored when proving Theorem 4.5.

For a standard basis element $v(\mathbf{I})$ of $1 \leq k \leq N-1$ wedges corresponding to an increasing sequence $\mathbf{I} = \{1 \leq i_1 < \cdots < i_k \leq N\}$, the next lemma defines a certain monomial $a(\hat{\mathbf{I}})$ that projects $v(\mathbf{I})$ onto $(-1)^{k-1}q\, v(\mathbf{I})$ and sends $v(\mathbf{I}')$ to zero for $\mathbf{I}' \neq \mathbf{I}$. Before stating the result, we give an example to demonstrate in the graphical description how this projector will be defined.

Example 4.7 Let $N = 8$, and consider the particle configuration $v(\mathbf{I}) = v_1 \wedge v_5 \wedge v_6$. With $a(\widehat{1\ 5\ 6}) = (a_0a_7)\cdot(a_4a_3a_2)\cdot(a_1a_5a_6)$ we obtain $a(\widehat{1\ 5\ 6})\cdot v_1\wedge v_5\wedge v_6 = (-1)^2 q\cdot v_1\wedge v_5\wedge v_6$, which looks as follows in the graphical description (Fig. 4).

The factor $a_1a_5a_6$ moves every particle one step forward clockwise. It is critical that we start by moving the particle at position 6 before moving the particle at position 5, as otherwise the result would be zero. But since there is a 'gap' at position 7, we can move the particle from site 6 to 7, and afterwards the particle from site 5 to 6, without obtaining zero. The assumption that $k < N$ ensures such a gap always exists.

After applying $a_1a_5a_6$, the particles are at positions 2, 6, and 7. The particle previously at position 5 is now at position 6, which is where we want a particle to be. The particle currently at position 2 can be moved to position 5 by applying the product $a_4a_3a_2$. The particle now at position 7 can be moved by a_0a_7 to position 1. Hence, the result of applying $(a_0a_7)\cdot(a_4a_3a_2)\cdot(a_1a_5a_6)$ is the same particle configuration as the original one. However, the answer must be multiplied by $\pm q$, since applying a_0a_7 involves crossing the zero position once. To determine the sign, note from Definition 4.2 that $(a_0a_7)\cdot(a_4a_3a_2)\cdot(a_1a_5a_6)(v_1 \wedge v_5 \wedge v_6) = q \cdot v_5 \wedge v_6 \wedge v_1 = (-1)^2 q \cdot v_1 \wedge v_5 \wedge v_6$, so the sign is $+$.

Now we describe the general procedure:

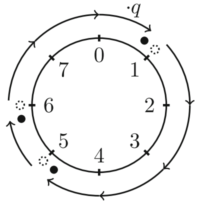

Fig. 4 The action of $a(\widehat{1\ 5\ 6})$ on the particle configuration $v_1 \wedge v_5 \wedge v_6$

Lemma 4.8 *Assume $v(\mathbf{I})$ is a particle configuration, where $\mathbf{I} = \{1 \le i_1 < \cdots < i_k \le N\}$ is an increasing sequence and $1 \le k \le N-1$. Then there exists an index ℓ such that $i_\ell + 1 < i_{\ell+1}$ (or $i_k + 1 < i_1$), i.e. the sequence has a 'gap' between i_ℓ and $i_{\ell+1}$. Split the sequence $\mathbf{I}$ into the two parts $\{i_1 < \cdots < i_\ell\}$ and $\{i_{\ell+1} < \cdots < i_k\}$. Set*

$$a(\hat{\mathbf{I}}) := (a_{i_1-1}a_{i_1-2}\dots a_{i_k+2}a_{i_k+1}) \cdot \prod_{s=1}^{k-1}(a_{i_{s+1}-1}a_{i_{s+1}-2}\dots a_{i_s+2}a_{i_s+1})$$
$$\cdot (a_{i_{\ell+1}}a_{i_{\ell+2}}\dots a_{i_{k-1}}a_{i_k}) \cdot (a_{i_1}a_{i_2}\dots a_{i_{\ell-1}}a_{i_\ell}), \tag{*}$$

where the indices are modulo N in the factor $(a_{i_1-1}a_{i_1-2}\dots a_{i_k+2}a_{i_k+1})$. Then

$$a(\hat{\mathbf{I}})v(\mathbf{I}') = \begin{cases} (-1)^{k-1}q \cdot v(\mathbf{I}) & \text{if } \mathbf{I}' = \mathbf{I}, \\ 0 & \text{for all } \mathbf{I}' \ne \mathbf{I} \text{ (of any length),} \end{cases}$$

and $a(\hat{\mathbf{I}})$ has $\mathbb{Z}^N$-degree $(1, 1, \dots, 1)$.

Proof The assertions can be seen using the graphical realization of V. The terms in the second line of equation (*) move a particle at site $i_j \in \mathbf{I}$ one step forward to $i_j + 1$ for each j, while the terms in the first line send the particle from $i_j + 1$ to the original position of i_{j+1}.

Consider first $a(\hat{\mathbf{I}})v(\mathbf{I})$. By applying $(a_{i_{\ell+1}}a_{i_{\ell+2}}\dots a_{i_{k-1}}a_{i_k}) \cdot (a_{i_1}a_{i_2}\dots a_{i_{\ell-1}}a_{i_\ell})$, every particle is first moved clockwise by one position. By our choice of the index i_ℓ, we avoid mapping the whole particle configuration to zero. After that step, every particle is moved by one of the factors $(a_{i_{s+1}-1}a_{i_{s+1}-2}\dots a_{i_s+2}a_{i_s+1})$ to the original position of its successor in the sequence $\mathbf{I}$, so the particle configuration remains the same. One of the particles has passed the zero position, so we have to multiply by $\pm q$. Definition 4.2 tells us the appropriate sign is $(-1)^{k-1}$.

Now consider $a(\hat{\mathbf{I}})v(\mathbf{I}')$ for $\mathbf{I}' \ne \mathbf{I}$. The monomial $(a_{i_{\ell+1}}a_{i_{\ell+2}}\dots a_{i_{k-1}}a_{i_k}) \cdot (a_{i_1}a_{i_2}\dots a_{i_{\ell-1}}a_{i_\ell})$ expects a particle at each of the sites $i_1, \dots, i_k$, so if any of these positions is empty in $v(\mathbf{I}')$, the result of applying $a(\hat{\mathbf{I}})$ is zero. If the positions $i_1, \dots, i_k$ are already filled, and there is an additional particle somewhere, multiplication by $(a_{i_{\ell+1}-1}a_{i_{\ell+1}-2}\dots a_{i_\ell+2}a_{i_\ell+1})$ will cause two particles to be at the same position, hence the result is again zero.

Since every a_j appears in $a(\hat{\mathbf{I}})$ exactly once, the monomial $a(\hat{\mathbf{I}})$ has $\mathbb{Z}^N$-degree $(1, 1, \dots, 1)$. □

Example 4.9 In the previous example, $N = 8$, $\mathbf{I} = (1, 5, 6)$, and we may assume the two subsequences are (1) and (5, 6). Then the terms in the second line of (*) are $(a_5a_6) \cdot (a_1) = a_1a_5a_6$. The term corresponding to $j = 1$ in the product on the first line of (*) is $a_4a_3a_2$, and the expression corresponding to $j = 2$ is empty, hence taken to be 1. The first factor on the first line is a_0a_7. Thus, for $\mathbf{I} = (1, 5, 6)$, $a(\hat{\mathbf{I}}) = (a_0a_7) \cdot (a_4a_3a_2) \cdot (a_1a_5a_6)$, as in Example 4.7. If the gap between 6 and 0 is used instead, the right-hand factor of the second line is $a_1a_5a_6$ and the left-hand factor is 1. The factors in the first line remain the same, and so one obtains the same expression for $a(\hat{\mathbf{I}})$.

Remark 4.10 Because V is a faithful module, $a(\hat{\mathbf{I}})$ is, as an element in $\widehat{\mathrm{nTL}}_N$ (i.e. up to reordering according to the defining relations), uniquely determined by the increasing sequence $\mathbf{I}$. One can read off $\mathbf{I}$ from $a(\hat{\mathbf{I}})$ as follows: In the defining equation (*) of $a(\hat{\mathbf{I}})$, the factors in the first line are pairwise commuting. The underlying subsequence $(i_{s+1}-1, i_{s+1}-2, \dots, i_s+2, i_s+1)$ corresponding to the factor $a_{i_{s+1}-1}a_{i_{s+1}-2}\dots a_{i_s+2}a_{i_s+1}$ of $a(\hat{\mathbf{I}})$ is a decreasing sequence. After all such decreasing sequences are removed from $a(\hat{\mathbf{I}})$, what remains is a product of generators a_j with an increasing subsequence of indices or a product of two such subsequences corresponding to the factors in the second line. This is $\mathbf{I}$. Given any monomial $a(\underline{r})$ of $\mathbb{Z}^N$-degree $(1,\dots,1)$, one can rewrite it using the relations in $\mathrm{n}\widehat{\mathrm{TL}}_N$ so that it is of the form $a(\hat{\mathbf{I}})$ for some increasing sequence $\mathbf{I}$. Then $v(\mathbf{I})$ is the unique standard basis element upon which $a(\underline{r}) = a(\hat{\mathbf{I}})$ acts by multiplication by $\pm q$.

5 Description of the center

In this section, we give an explicit description of the center C_N of $\mathrm{n}\widehat{\mathrm{TL}}_N$. We start with the following initial characterisation of the central elements:

Lemma 5.1 *Any central element c in $\mathrm{n}\widehat{\mathrm{TL}}_N$ with constant term 0 is a linear combination of monomials $a(\underline{j}) = a_{j_1} \cdot \dots \cdot a_{j_m}$ where every generator a_i, $0 \le i \le N-1$, appears at least once. In particular, a homogeneous nonconstant central element c has $\mathbb{Z}$-degree at least N.*

Proof Assume $c = \sum_{\underline{j}} c_{\underline{j}} a(\underline{j})$, where $c_{\underline{j}} \in \Bbbk$ for all $\underline{j}$. By Lemma 3.1, we can assume c is a homogeneous central element with respect to the $\mathbb{Z}^N$-grading. By our assumption, $c \notin \Bbbk$. For all i, we need to show that a_i occurs in each monomial $a(\underline{j})$ appearing in c. Without loss of generality, we show this for $i = 0$. If some summand is missing a_0, then no summand contains a_0 because c is homogeneous. Hence $a_0 a(\underline{j}) \neq 0$ and $a(\underline{j}) a_0 \neq 0$ for all $\underline{j}$ with $c_{\underline{j}} \neq 0$, and since $a_0 c = c a_0$, none of the $a(\underline{j})$ can contain the factor a_1 either, as otherwise the factor a_0 cannot pass through c from left to right (so also a_{N-1} cannot be contained in the $a(\underline{j})$). Proceeding inductively, we see that all $a(\underline{j})$ must be a constant, contrary to our assumption. □

The next proposition states that on the standard wedge basis vector $v(\mathbf{I})$ of V, any central element acts via multiplication by a polynomial $p_k \in \Bbbk[q]$ that only depends on the length $k = |\mathbf{I}|$ of the increasing sequence $\mathbf{I} = \{1 \le i_1 < \dots < i_k \le N\}$. In other words, the decomposition of V into the summands $\Bbbk[q] \otimes \bigwedge^k \Bbbk^N$ is a decomposition with respect to different central characters (apart from the two trivial summands for $k \in \{0, N\}$).

Proposition 5.2 *For any central element $c \in \mathrm{n}\widehat{\mathrm{TL}}_N$ and all increasing sequences $\mathbf{I}$ with fixed length k, there is some element $p_k \in \Bbbk[q]$ such that $cv(\mathbf{I}) = p_k\, v(\mathbf{I})$.*

Proof We may assume c is a nonconstant $\mathbb{Z}^N$-homogeneous central element of $\mathrm{n}\widehat{\mathrm{TL}}_N$. For $k \in \{0, N\}$, the action of a generator a_i on a monomial of length k is 0, so $p_k = 0$ for such values of k. Now consider $1 \le k \le N-1$, and suppose that $\mathbf{I} = \{1 \le i_1 < \dots < i_k \le N\}$ is an increasing sequence of length k. According to Lemma 4.4 (b), the number of wedges in a vector remains constant under the action of the a_i. Hence $cv(\mathbf{I}) = \sum_{|\mathbf{I}'|=k} c_{\mathbf{I}'}\, v(\mathbf{I}')$ for some polynomials $c_{\mathbf{I}'} \in \Bbbk[q]$. We want to prove that $c_{\mathbf{I}'} = 0$ for all $\mathbf{I}' \neq \mathbf{I}$.

We have shown in Lemma 4.8 that to each increasing sequence $\mathbf{J} \subset \{1,\dots,N\}$ there corresponds a monomial $a(\hat{\mathbf{J}}) \in \mathrm{n}\widehat{\mathrm{TL}}_N$ that allows us to select a single basis vector:

$$a(\hat{\mathbf{J}})v(\mathbf{I}) \;=\; \begin{cases} (-1)^{k-1} q v(\mathbf{J}) & \text{if } \mathbf{I} = \mathbf{J}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for $\mathbf{J} \neq \mathbf{I}$, we see that

$$0 \;=\; c(a(\hat{\mathbf{J}})v(\mathbf{I})) \;=\; a(\hat{\mathbf{J}})(cv(\mathbf{I})) \;=\; a(\hat{\mathbf{J}}) \left(\sum_{|\mathbf{I}'|=k} c_{\mathbf{I}'}\, v(\mathbf{I}') \right) = c_{\mathbf{J}}\, (-1)^{k-1} q v(\mathbf{J}),$$

implying $c_{\mathbf{J}} = 0$ for $\mathbf{J} \neq \mathbf{I}$. Hence, we may assume for each increasing sequence $\mathbf{I}$ that $cv(\mathbf{I}) = p_{\mathbf{I}}\, v(\mathbf{I})$ for some polynomial $p_{\mathbf{I}} \in \Bbbk[q]$.

Now it is left to show that $p_{\mathbf{I}} = p_{\mathbf{I}'}$ for all $\mathbf{I}'$ with $|\mathbf{I}'| = |\mathbf{I}| = k$. It is enough to verify this for $\mathbf{I}, \mathbf{I}'$ that differ in exactly one entry, i.e. $i_s = i$, $i'_s = i+1$, and $i_\ell = i'_\ell$ for all $\ell \neq s$, for some $1 \leq s \leq k$ and $i \in \mathbb{Z}/N\mathbb{Z}$. If $1 \leq i \leq N-1$, we have

$$p_{\mathbf{I}'}\, v(\mathbf{I}') = cv(\mathbf{I}') = c(a_i v(\mathbf{I})) = a_i(cv(\mathbf{I})) = a_i(p_{\mathbf{I}}\, v(\mathbf{I})) = p_{\mathbf{I}}\, v(\mathbf{I}'),$$

and if $i = 0$, we get

$$\begin{aligned}(-1)^{k-1} q p_{\mathbf{I}'}\, v(\mathbf{I}') = (-1)^{k-1} q cv(\mathbf{I}') &= c(a_0 v(\mathbf{I})) = a_0(cv(\mathbf{I})) = a_0(p_{\mathbf{I}}\, v(\mathbf{I})) \\ &= (-1)^{k-1} q p_{\mathbf{I}}\, v(\mathbf{I}').\end{aligned}$$

Hence, $p_{\mathbf{I}'} = p_{\mathbf{I}}$, and this common polynomial is the desired polynomial p_k. □

Corollary 5.3 *Any central element in $n\widehat{TL}_N$ with constant term* 0 *acts on a standard basis vector $v(\mathbf{I}) \in \mathsf{V}$ as multiplication by an element of $q\Bbbk[q]$.*

Proof According to Lemma 5.1, each summand of such a central element must contain the factor a_0, and a_0 acts on a wedge product by 0 or multiplication by $\pm q$. □

Now we are ready to introduce nontrivial central elements in $\mathrm{n}\widehat{\mathrm{TL}}_N$. For each $1 \leq k \leq N-1$, set

$$\mathbf{t}_k := (-1)^{k-1} \sum_{|\mathbf{I}|=k} a(\hat{\mathbf{I}}), \tag{1}$$

where the monomials $a(\hat{\mathbf{I}})$ correspond to increasing sequences $\mathbf{I} = \{1 \leq i_1 < \cdots < i_k \leq N\}$ of length k as defined in Lemma 4.8.

Example 5.4 In $\mathrm{n}\widehat{\mathrm{TL}}_3$:

$$\begin{aligned}\mathbf{t}_1 &= a_2a_1a_0 + a_0a_2a_1 + a_1a_0a_2,\\ \mathbf{t}_2 &= -a_0a_1a_2 - a_1a_2a_0 - a_2a_0a_1.\end{aligned}$$

In $\mathrm{n}\widehat{\mathrm{TL}}_4$:

$$\begin{aligned}\mathbf{t}_1 &= a_3a_2a_1a_0 + a_0a_3a_2a_1 + a_1a_0a_3a_2 + a_2a_1a_0a_3,\\ \mathbf{t}_2 &= -a_0a_3a_1a_2 - a_0a_2a_1a_3 - a_3a_2a_0a_1 - a_1a_0a_2a_3 - a_1a_3a_0a_2 - a_2a_1a_3a_0\\ \mathbf{t}_3 &= a_0a_1a_2a_3 + a_1a_2a_3a_0 + a_2a_3a_0a_1 + a_3a_0a_1a_2.\end{aligned}$$

In the graphical realization of V, $\mathbf{t}_k$ acts by annihilating all particle configurations whose number of particles is different from k. For particle configurations having k particles, every particle is moved clockwise to the original site of the next particle. Hence, the particle configuration itself remains fixed by the action of $\mathbf{t}_k$ (and it is multiplied with $(-1)^{2(k-1)}q = q$, since a particle has been moved through position 0). All the $\mathbf{t}_k$ have $\mathbb{Z}^N$-degree equal to $(1, \ldots, 1)$ and $\mathbb{Z}$-degree equal to N. Any monomial whose $\mathbb{Z}^N$-degree is $(1, \ldots, 1)$ occurs as a summand in some central element (after possibly reordering the factors), and the number of summands of $\mathbf{t}_k$ equals $\binom{N}{k} = \dim(\bigwedge^k \Bbbk^N)$; see Remark 4.10.

Theorem 5.5 *1. The $\mathbf{t}_k$ are central for all $1 \leq k \leq N-1$, and the center of $n\widehat{TL}_N$ is generated by* 1 *and the $\mathbf{t}_k$, $1 \leq k \leq N-1$.*

2. *The subalgebra generated by $\boldsymbol{t}_k$ is isomorphic to the polynomial ring $\Bbbk[q]$ for all $1 \leq k \leq N-1$. Moreover $\boldsymbol{t}_k\boldsymbol{t}_\ell = 0$ for all $k \neq \ell$. Hence the center of $n\widehat{TL}_N$ is the subalgebra*

$$\mathsf{C}_N \;=\; \Bbbk \oplus \boldsymbol{t}_1\Bbbk[\boldsymbol{t}_1] \oplus \cdots \oplus \boldsymbol{t}_{N-1}\Bbbk[\boldsymbol{t}_{N-1}] \;\cong\; \frac{\Bbbk[\boldsymbol{t}_1, \ldots, \boldsymbol{t}_{N-1}]}{(\boldsymbol{t}_k\boldsymbol{t}_\ell \mid k \neq \ell)}.$$

Proof 1. The action of $\mathbf{t}_k$ on V is the projection onto the $\mathrm{n}\widehat{\mathrm{TL}}_N$-submodule $\Bbbk[q] \otimes \bigwedge^k \Bbbk^N$ followed by multiplication by q. This commutes with the action of every other element of $\mathrm{n}\widehat{\mathrm{TL}}_N$. Since V is a faithful module, $\mathbf{t}_k$ commutes with any element of $\mathrm{n}\widehat{\mathrm{TL}}_N$. As we have seen in Proposition 5.2, any central element c without constant term acts on the summand $\Bbbk[q] \otimes \bigwedge^k \Bbbk^N$ via multiplication by some polynomial $p_k^c \in q\Bbbk[q]$. Once again using the faithfulness of V, we get that $c = \sum_{k=1}^{N-1} p_k^c(\mathbf{t}_k)$.

2. Recall that $\Bbbk[q] \otimes \bigwedge^k \Bbbk^N$ is a free $\Bbbk[q]$-module of rank $\binom{N}{k}$. Since $\mathbf{t}_k$ acts by multiplication with q on that module, the subalgebra of $\mathrm{n}\widehat{\mathrm{TL}}_N$ generated by $\mathbf{t}_k$ must be isomorphic to the polynomial ring $\Bbbk[q]$. Since $a(\hat{\mathbf{J}})a(\hat{\mathbf{I}}) = 0$ for all $\mathbf{J} \neq \mathbf{I}$, we get $\mathbf{t}_k\mathbf{t}_\ell = 0$ for $k \neq \ell$, as they consist of pairwise distinct summands. □

Theorem 5.5 enables us to describe the $\Bbbk$-algebra $\mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N}(\mathsf{W})$ of $\mathrm{n}\widehat{\mathrm{TL}}_N$-endomorphisms of the space of nontrivial particle configurations $\mathsf{W} := \bigoplus_{k=1}^{N-1}\left(\Bbbk[q] \otimes \bigwedge^k \Bbbk^N\right) \subset \mathsf{V}$. We first observe that on W multiplication by q is given by the action of a central element in C_N, therefore it is justified to speak about $\Bbbk[q]$-linearity of a $\mathrm{n}\widehat{\mathrm{TL}}_N$-endomorphism of W.

Lemma 5.6 $\mathrm{End}_{n\widehat{TL}_N}(\mathsf{W}) \subset \mathrm{End}_{\Bbbk[q]}(\mathsf{W})$*, hence any $n\widehat{TL}_N$-module endomorphism φ of W is $\Bbbk[q]$-linear.*

Proof Observe that $\sum_{k=1}^{N-1} \mathbf{t}_k \in \mathrm{n}\widehat{\mathrm{TL}}_N$ acts by multiplication by q on every element in W. Therefore multiplication by q commutes with the application of every $\varphi \in \mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N}(\mathsf{W})$. □

Proposition 5.7 *The endomorphism algebra $\mathrm{End}_{n\widehat{TL}_N}(\mathsf{W})$ is isomorphic to a direct sum of $N-1$ polynomial algebras $\Bbbk[T_1] \oplus \cdots \oplus \Bbbk[T_{N-1}]$.*

Proof The proof is very similar to that of Proposition 5.2. First we show that $\varphi(v(\mathbf{I}))$ is a $\Bbbk[q]$-linear multiple of $v(\mathbf{I})$ for any $\varphi \in \mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N}(\mathsf{W})$ and any increasing sequence $\mathbf{I}$. This statement holds if and only if $\pm q\varphi(v(\mathbf{I})) \in \Bbbk[q]\, v(\mathbf{I})$. Indeed, by Lemmas 4.8 and 5.6 we get

$$\pm q\varphi(v(\mathbf{I})) \;=\; \varphi(\pm q v(\mathbf{I})) \;=\; \varphi(a(\hat{\mathbf{I}})v(\mathbf{I})) \;=\; a(\hat{\mathbf{I}})\varphi(v(\mathbf{I})) \;\in \Bbbk[q]\, v(\mathbf{I}).$$

Therefore, we can write $\varphi(v(\mathbf{I})) = p_{\mathbf{I}} \cdot v(\mathbf{I})$ for some polynomial $p_{\mathbf{I}} \in \Bbbk[q]$. Note that this implies

$$\mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N}\left(\bigoplus_{k=1}^{N-1}\left(\Bbbk[q] \otimes \bigwedge^k \Bbbk^N\right)\right) \;=\; \bigoplus_{k=1}^{N-1}\left(\mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N}\left(\Bbbk[q] \otimes \bigwedge^k \Bbbk^N\right)\right).$$

What remains is to show that these polynomials only depend on the number of particles in $\mathbf{I}$, in other words there exists $p_k \in \Bbbk[q]$ so that $p_{\mathbf{I}} = p_k$ for all $\mathbf{I}$ with $|\mathbf{I}| = k$. Again it suffices to show this for two sequences $\mathbf{I}, \mathbf{I}'$ of length k that differ in exactly one entry. So say $i_s = i$, $i'_s = i+1$, and $i_\ell = i'_\ell$ for all $\ell \neq s$, for some $1 \leq s \leq k$ and $i \in \mathbb{Z}/N\mathbb{Z}$. When $1 \leq i \leq N-1$,

$$p_{\mathbf{I}'}\, v(\mathbf{I}') \;=\; \varphi(v(\mathbf{I}')) \;=\; \varphi(a_i v(\mathbf{I})) \;=\; a_i\varphi(v(\mathbf{I})) \;=\; a_i(p_{\mathbf{I}}\, v(\mathbf{I})) \;=\; p_{\mathbf{I}}\, v(\mathbf{I}'),$$

and when $i = 0$,

$$(-1)^{k-1} q p_{\mathbf{I}'}\, v(\mathbf{I}') = (-1)^{k-1} q \varphi(v(\mathbf{I}')) = \varphi(a_0 v(\mathbf{I})) = a_0 \varphi(v(\mathbf{I})) = a_0(p_{\mathbf{I}}\, v(\mathbf{I}))$$
$$= (-1)^{k-1} q p_{\mathbf{I}}\, v(\mathbf{I}').$$

Hence we can write $\varphi = \sum_{k=1}^{N-1} p_k \pi_k$ where π_k is the projection onto $\Bbbk[q] \otimes \bigwedge^k \Bbbk^N$, and we get that

$$\mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N} \left(\Bbbk[q] \otimes \textstyle\bigwedge^k \Bbbk^N\right) = \Bbbk[T_k],$$

where T_k denotes the multiplication action of the central element $\mathbf{t}_k$, which is indeed a $\mathrm{n}\widehat{\mathrm{TL}}_N$-module endomorphism of W. Thus, $\mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N}(\mathsf{W})$ is isomorphic to a direct sum of polynomial algebras as claimed. □

Remark 5.8 The arguments in the proof of Proposition 5.7 remain valid even if we specialize the indeterminate q to some element in $\Bbbk \setminus \{0\}$. In this case, we obtain that the summands $\bigwedge^k \Bbbk^N$ are simple modules and $\mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N}\left(\bigoplus_{k=1}^{N-1} \bigwedge^k \Bbbk^N\right) \cong \Bbbk^{N-1}$. For $q = 0$, the situation is more complicated: If q is specialized to zero, the generator a_0 acts by zero on the module. The action of $\mathrm{n}\widehat{\mathrm{TL}}_N$ factorizes over nTL_N, and the module $\bigwedge^k \Bbbk^N$ is no longer simple. Instead it has a one-dimensional head spanned by the particle configuration $v(1, \ldots, k)$, and any endomorphism is given by choosing an image of this top configuration. It is always possible to map it to itself and to the one-dimensional socle spanned by $v(N-k, \ldots, N)$, but in general there are more endomorphisms. For example, in $\bigwedge^4 \Bbbk^8$, the image of $v(1, 2, 3, 4)$ may be any linear combination of $v(1, 2, 3, 4)$, $v(2, 3, 4, 8)$, $v(3, 4, 7, 8)$, $v(4, 6, 7, 8)$ and $v(5, 6, 7, 8)$, so that $\mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_8}\left(\bigwedge^4 \Bbbk^8\right)$ is 5-dimensional.

6 The affine nilTemperley–Lieb algebra is finitely generated over its center

The affine nilTemperley–Lieb algebra is infinite dimensional when $N \geq 3$; however, the following finiteness result holds:

Theorem 6.1 *The algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$ is finitely generated over its center.*

Proof Given an arbitrary monomial $a(\underline{j}) \in \mathrm{n}\widehat{\mathrm{TL}}_N$, we first factor it as $a(\underline{j}') \cdot a(\underline{j}^{(0)})$ in the following way: Take the minimal particle configuration $\mathbf{J} = \{1 \leq j_1 < \cdots < j_k \leq N\}$ on which the monomial $a(\underline{j})$ acts nontrivially; see Remark 4.3. The monomial $a(\underline{j})$ moves all of the particles by at least one step, because the particle configuration was assumed to be minimal. Using the faithfulness of the representation, we know that we may reorder the monomial $a(\underline{j})$ so that first each particle is moved one step clockwise, and afterwards the remaining particle moves are carried out. Hence, we may choose some factorization $a(\underline{j}) = a(\underline{j}') \cdot a(\underline{j}^{(0)})$, where $\underline{j}^{(0)}$ is a sequence obtained by permuting $j_1, \ldots, j_k$ so that the particle at position j_r is moved one step clockwise by the action of a_{j_r} for all $1 \leq r \leq k$. The remaining particle moves are carried out by $a(\underline{j}')$.

In Sect. 8, this decomposition is explicitly constructed (not using the faithful representation).

Next, we want to find an expression of the form

$$a(\underline{j}) = a_{\mathrm{fin}} \cdot \mathbf{t}_k^n \cdot a(\underline{j}^{(0)}),$$

where a_{fin} is a monomial of some subalgebra ${}^i\text{nTL}_N$ of $\text{n}\widehat{\text{TL}}_N$, $\mathbf{t}_k^n$ is in the center of $\text{n}\widehat{\text{TL}}_N$, and $a(\underline{j}^{(0)})$ is the above factor. Here

$$ {}^i\text{nTL}_N = \langle a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_{N-1} \rangle \tag{2} $$

denotes a copy of the finite nilTemperley–Lieb algebra nTL_N sitting in $\text{n}\widehat{\text{TL}}_N$. To accomplish this, we have to subdivide the action of $a(\underline{j})$ on the particle configuration $\mathbf{J} = \{j_1 < \cdots < j_k\}$ one more time. There are two cases:

1. There is an index i not appearing in $\underline{j}'$. In this case, $a(\underline{j}')$ is an element of ${}^i\text{nTL}_N$ and we are done.
2. All indices appear at least $n \geq 1$ times in $\underline{j}'$. Let us investigate the action of $a(\underline{j}')$ on the particle configuration $v(\mathbf{I}) = a(\underline{j}^{(0)})v(\mathbf{J})$, where $\mathbf{I} = \{j_1 + 1, \dots, j_k + 1\}$. Note that $\mathbf{I}$ is the minimal particle configuration for $a(\underline{j}')$. Each of the particles in $\mathbf{I}$ is moved by $a(\underline{j}')$ to the position of the next particle in the sequence $\mathbf{I}$, because there is no index missing (a missing index is equivalent to a particle being stopped before reaching the position of its successor), before possibly continuing to move along the circle. Again invoking the faithfulness of the representation, we can rewrite $a(\underline{j}') = a(\underline{j}'') \cdot a(\hat{\mathbf{I}})^n$, with the monomial $a(\hat{\mathbf{I}})$ from Lemma 4.8. For maximal n, the remaining factor $a(\underline{j}'')$ is an element of ${}^i\text{nTL}_N$ for some i. Observe that $a(\hat{\mathbf{I}})^n a(\underline{j}^{(0)}) = \mathbf{t}_k^n a(\underline{j}^{(0)})$, which follows immediately from the definition of $\mathbf{t}_k$ and Lemma 4.8.

Therefore, we have shown that

$$ a(\underline{j}) = a(\underline{j}') \cdot a(\underline{j}^{(0)}) = a_{\text{fin}} \cdot a(\hat{\mathbf{I}})^n \cdot a(\underline{j}^{(0)}) = a_{\text{fin}} \cdot \mathbf{t}_k^n \cdot a(\underline{j}^{(0)}), $$

where $n = 0$ in the first case. Since there are only finitely many monomials in ${}^0\text{nTL}_N, {}^1\text{nTL}_N, \dots, {}^{N-1}\text{nTL}_N$ and only finitely many monomials $a(\underline{j}^{(0)})$ such that every index $0, 1, \dots, N-1$ occurs at most once in the sequence $a(\underline{j}^{(0)})$, the affine nilTemperley–Lieb algebra is indeed finitely generated over its center. □

Remark 6.2 The affine nilTemperley–Lieb algebra is not free over its center (see [18]).

7 Embeddings of affine nilTemperley–Lieb algebras

In the proof of Theorem 6.1, we have used the N obvious embeddings of nTL_N into $\text{n}\widehat{\text{TL}}_N$ coming from the N different embeddings of the Coxeter graph A_{N-1} into $\tilde{\mathsf{A}}_{N-1}$. Next we construct N embeddings of $\text{n}\widehat{\text{TL}}_N$ into $\text{n}\widehat{\text{TL}}_{N+1}$. They correspond to the subdivision of an edge of $\tilde{\mathsf{A}}_{N-1}$ by inserting a vertex on the edge to obtain $\tilde{\mathsf{A}}_N$.

Theorem 7.1 *Let $N \geq 3$. For any number $0 \leq m \leq N-1$, there is a unital embedding of algebras $\varepsilon_m : n\widehat{TL}_N \to n\widehat{TL}_{N+1}$ given by*

$$ a_i \mapsto \begin{cases} a_i & \text{for } 0 \leq i \leq m-1, \\ a_{m+1}a_m & \text{for } i = m, \\ a_{i+1} & \text{for } m+1 \leq i \leq N-1. \end{cases} \tag{3} $$

Lemma 7.2 *For $N \geq 3$, the map ε_m from $n\widehat{TL}_N$ to $n\widehat{TL}_{N+1}$ given by (3) is an algebra homomorphism.*

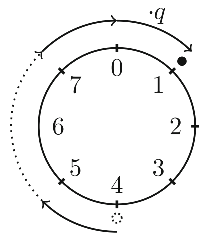

Fig. 5 $\varepsilon_5(\mathrm{n}\widehat{\mathrm{TL}}_7) \subset \mathrm{n}\widehat{\mathrm{TL}}_8$: The action of $\varepsilon_5(a_0a_6a_5a_4) = \tilde{a}_0\tilde{a}_7\tilde{a}_6\tilde{a}_5\tilde{a}_4$ on the particle configuration $v(4)$

Proof Due to the circular nature of the relations, it suffices to check this for ε_0. This amounts to showing the following, since all the other relations are readily apparent. To avoid confusion, we indicate generators of $\mathrm{n}\widehat{\mathrm{TL}}_{N+1}$ in these calculations by $\tilde{a}_i$:

$$(\tilde{a}_1\tilde{a}_0)(\tilde{a}_1\tilde{a}_0) = \tilde{a}_1(\tilde{a}_0\tilde{a}_1\tilde{a}_0) = 0, \quad \tilde{a}_2(\tilde{a}_1\tilde{a}_0)\tilde{a}_2 = (\tilde{a}_2\tilde{a}_1\tilde{a}_2)\tilde{a}_0 = 0, \quad \tilde{a}_N(\tilde{a}_1\tilde{a}_0)\tilde{a}_N = \tilde{a}_1(\tilde{a}_N\tilde{a}_0\tilde{a}_N) = 0,$$
$$(\tilde{a}_1\tilde{a}_0)\tilde{a}_2(\tilde{a}_1\tilde{a}_0) = (\tilde{a}_1\tilde{a}_2)(\tilde{a}_0\tilde{a}_1\tilde{a}_0) = 0, \quad (\tilde{a}_1\tilde{a}_0)\tilde{a}_N(\tilde{a}_1\tilde{a}_0) = (\tilde{a}_1\tilde{a}_0\tilde{a}_1)(\tilde{a}_N\tilde{a}_0) = 0.$$

Remark 7.3 How should one visualize the action of $\varepsilon_m(\mathrm{n}\widehat{\mathrm{TL}}_N) \subset \mathrm{n}\widehat{\mathrm{TL}}_{N+1}$ on the particle configurations on a circle with $N+1$ positions? Except for a_m, all generators of $\mathrm{n}\widehat{\mathrm{TL}}_N$ are mapped to corresponding generators of $\mathrm{n}\widehat{\mathrm{TL}}_{N+1}$. They will act as before, by moving a particle one step clockwise around the circle. Since a_m is mapped by ε_m to the product $\tilde{a}_{m+1}\tilde{a}_m$ in $\mathrm{n}\widehat{\mathrm{TL}}_{N+1}$, it will move a particle from m to $m+2$ as depicted in Fig. 5. In other words, the elements in $\varepsilon_m(\mathrm{n}\widehat{\mathrm{TL}}_N)$ do not move a particle to or from position $m+1$.

Next we introduce a basis of $\mathrm{n}\widehat{\mathrm{TL}}_N$ that will enable us to see directly that these homomorphisms are embeddings. The basis has a simple description in terms of the graphical representation V from Sect. 4. For any two particle configurations with $1 \le k \le N-1$ particles corresponding to the increasing sequences $\mathbf{I} = \{1 \le i_1 < \cdots < i_k \le N\}$ and $\mathbf{J} = \{1 \le j_1 < \cdots < j_k \le N\}$, there is a monomial in $\mathrm{n}\widehat{\mathrm{TL}}_N$ moving particles at the positions $\mathbf{J}$ to the positions $\mathbf{I}$. We require that every particle from $\mathbf{J}$ be moved by at least one step, but we do not prescribe explicitly which of the j's is mapped to which of the i's. For $\mathbf{I} \neq \mathbf{J}$, take $e_{\mathbf{IJ}}$ to be the monomial such that the power of q in $e_{\mathbf{IJ}}v(\mathbf{J}) = \pm q^\ell v(\mathbf{I})$ is minimal (under the assumption that every particle from $\mathbf{J}$ must be moved). By faithfulness of the graphical representation, $e_{\mathbf{IJ}}$ is uniquely determined. For $\mathbf{I} = \mathbf{J}$, we have $e_{\mathbf{II}} = a(\hat{\mathbf{I}})$, the special monomial defined in Sect. 4, hence $e_{\mathbf{II}}v(\mathbf{I}) = \pm q v(\mathbf{I})$. Observe that one can write $\mathbf{t}_k = \sum_{|\mathbf{I}|=k} e_{\mathbf{II}}$, where the sum runs over all possible increasing sequences $\mathbf{I}$ of length k, and that $\mathbf{t}_k^\ell e_{\mathbf{IJ}}$ is a monomial, since all but one summand vanish for $k = |\mathbf{I}|$.

Remark 7.4 The condition that $e_{\mathbf{IJ}}$ move all particles from $\mathbf{J}$ by at least one step guarantees that it acts as zero on all particle configurations with fewer particles than $|\mathbf{I}| = |\mathbf{J}|$.

For example, when $N = 7$,

$$e_{(2)(1)} = a_1, \quad e_{(0,2)(0,1)} = a_6a_5a_4a_3a_1a_2a_0a_1.$$

(Note that a_1 moves $v(0,1)$ to $v(0,2)$, but this does not satisfy the requisite property that all the particles must be moved by at least one step.) If we apply the factorization of monomials from Theorem 6.1 to $e_{\mathbf{IJ}}$, the minimality condition implies that $e_{\mathbf{IJ}} = a_{\mathrm{fin}} \cdot 1 \cdot a(\underline{j}^{(0)})$, where if $\mathbf{J} = \{j_1 < \cdots < j_k\}$, then $\underline{j}^{(0)}$ is a sequence obtained by permuting the elements of $\mathbf{J}$.

Theorem 7.5 *The set of monomials*

$$\{1\} \cup \{\mathbf{t}_k^\ell e_{\mathbf{IJ}} \mid \ell \in \mathbb{Z}_{\ge 0},\ 1 \le |\mathbf{I}| = |\mathbf{J}| = k \le N-1\}$$

defines a $\Bbbk$-basis of the affine nilTemperley–Lieb algebra $\widehat{nTL}_N$.

Proof First, observe that $\mathbf{t}_k^\ell e_{\mathbf{IJ}}$ is indeed a monomial since $|\mathbf{I}| = k$. We show that the elements $\mathbf{t}_k^\ell e_{\mathbf{IJ}}$ act $\Bbbk$-linearly independently on the graphical representation $\mathsf{V} = \bigoplus_{k=0}^N \left(\Bbbk[q] \otimes \bigwedge^k \Bbbk^N\right)$. By Remark 7.4, the monomial $e_{\mathbf{IJ}}$ acts by zero on summands $\Bbbk[q] \otimes \bigwedge^{k'} \Bbbk^N$ for $k' < |\mathbf{I}|$. On $\Bbbk[q] \otimes \bigwedge^{|\mathbf{I}|} \Bbbk^N$, the matrix representing the action of $\mathbf{t}_k^\ell e_{\mathbf{IJ}}$ relative to the standard basis has exactly one nonzero entry, and this one distinguishes all monomials with the same minimal number of particles $|\mathbf{I}| = |\mathbf{J}|$. From these two observations, the linear independence follows. On the other hand, given any nonzero monomial in $\widehat{\mathrm{nTL}}_N$, there exists a minimal particle configuration $\mathbf{J}$ on which it acts nontrivially. Recording the image particle configuration $\mathbf{I}$ and the power of q, we conclude that there is some ℓ so that the element $\mathbf{t}_k^\ell e_{\mathbf{IJ}}$ acts on V in the same way as the given monomial does. Due to the faithfulness of this representation (see Theorem 4.5), the proposition follows. □

In Sect. 8, a basis is constructed using a different approach (without relying on the faithful representation). Both bases are labelled by pairs of particle configurations (pairs of increasing sequences) together with a natural number ℓ. Up to an index shift in the output configuration $\mathbf{I}$ and a shift of the natural number ℓ, the labelling sets agree, and both bases actually coincide.

Proof (Theorem 7.1) We have already proven in Lemma 7.2 that ε_m is an algebra homomorphism. Using Remark 7.3, observe that the monomial $e_{\mathbf{IJ}} \in \widehat{\mathrm{nTL}}_N$ is mapped to a monomial $\tilde{e}_{\mathbf{I'J'}} \in \widehat{\mathrm{nTL}}_{N+1}$ (tilde again indicates in $\widehat{\mathrm{nTL}}_{N+1}$), where the new index sets are obtained by $i \mapsto i$ for $0 \leq i \leq m$ and $i \mapsto i+1$ for $m+1 \leq i \leq N-1$. The injectivity follows since basis elements $\left(\sum_{|\mathrm{K}|=k} e_{\mathrm{KK}}\right)^\ell \cdot e_{\mathbf{IJ}}$ of $\widehat{\mathrm{nTL}}_N$ are mapped to basis elements $\left(\sum_{|\mathrm{K}'|=k} \tilde{e}_{\mathrm{K}'\mathrm{K}'}\right)^\ell \cdot \tilde{e}_{\mathbf{I'J'}}$ of $\widehat{\mathrm{nTL}}_{N+1}$. □

Remark 7.6 It is possible to verify this theorem on generators and relations in the language of Sect. 8 without using the graphical description.

Remark 7.7 Observe that these embeddings work specifically for the affine *nil*Temperley–Lieb algebras but fail for the ordinary Temperley–Lieb algebras. The relation that fails to hold is the braid relation for Temperley–Lieb algebras, i.e. $a_i a_{i\pm1} a_i = a_i$. Interestingly, the relation $a_i^2 = \delta a_i$ is respected for $\delta = 1$.

8 A normal form and the faithfulness of the graphical representation

In this section, we prove Theorem 4.5 which we recall here:

Proposition *For $N \geq 3$, V is a faithful $\widehat{nTL}_N$-module with respect to the action described in Definition 4.2.*

For the proof, we will explicitly prove the linear independence of the matrices representing the monomials in $\widehat{\mathrm{nTL}}_N$. We proceed in three steps: (1) First, we define a normal form for the monomials. (2) Next, we find a bijection between the monomials and certain pairs of particle configurations together with a power of q. In other words, we find a basis for $\widehat{\mathrm{nTL}}_N$ and describe a labeling set. (3) The final step is the description of the action of a monomial on V using its matrix realization. The matrices representing the monomials have a distinguished nonzero entry that is given in terms of the particle configurations and the power of q from the bijection, and for most matrices, this is the only nonzero entry. From this description it will quickly follow that all these matrices are linearly independent.

8.1 Some useful facts

The following lemma characterises nonzero monomials in $\mathrm{n}\widehat{\mathrm{TL}}_N$. They correspond to fully commutative elements in $\widehat{\mathrm{TL}}_N$, see [9].

Lemma 8.1 *The monomial $a(\underline{j}) \neq 0$ if and only if for any two neighbouring appearances of a_i in $a(\underline{j})$ there are exactly one a_{i+1} and one a_{i-1} in between, apart from possible factors a_ℓ for $\ell \neq i-1, i, i+1$ (indices to be understood modulo N).*

According to this result, two consecutive a_i have to enclose a_{i+1} and a_{i-1}, i.e. $a_i \dots a_{i\pm1} \dots a_{i\mp1} \dots a_i$, with the dots being possible products of a_ℓ's with $\ell \neq i \pm 1, i$. This lemma is a special case of [9, Lem. 2.6]; here is a quick proof for the convenience of the reader.

Proof The monomial $a(\underline{j})$ is zero if and only if we can bring two neighbouring factors a_i together so that we obtain either a_i^2 ('square') or $a_i a_{i\pm1} a_i$ ('braid'). But expressions of the form $a_i \dots a_{i\pm1} \dots a_{i\mp1} \dots a_i$ cannot be resolved this way by commutativity relations. On the other hand, if there are two neighbouring factors a_i with either none or only one of the terms $a_{i\pm1}$ in between, we get after commutations either a_i^2 or $a_i a_{i\pm1} a_i$. If there are at least two factors a_{i+1} (or a_{i-1}) in between the two a_i, one can repeat the argument: Either we can create a square or a braid, or we have at least two factors of the same kind in between. In the case of a square or a braid we are done; otherwise we pick two neighbouring a_{i+k} in the kth step of the argument. Since we always consider the space in between two neighbouring factors $a_i, a_{i+1}, \dots, a_{i+k}$, none of the previous $a_i, a_{i+1}, \dots, a_{i+k-1}$ occurs between the two neighbouring a_{i+k}. Unless we found a square or a braid in an earlier step, we end up in step $N-1$ with a subexpression of the form $a_r a_{r\pm1}^m a_r$ which is zero for any $m \geq 0$. □

Definition 8.2 For any $i \in \{0, 1, \dots, N-1\}$, we define a (clockwise) order $\overset{i}{\prec}$ on the set $\{0, 1, \dots, N-1\}$ starting at i by

$$i \overset{i}{\prec} i+1 \overset{i}{\prec} \dots \overset{i}{\prec} i+N-1.$$

8.2 Step 1: A normal form

Given an arbitrary nonzero monomial $a(\underline{j})$ in $\mathrm{n}\widehat{\mathrm{TL}}_N$, reorder its factors according to the following algorithm (as usual, the indices are considered modulo N):

1. Find all factors a_i in $a(\underline{j})$ with no a_{i-1} to their right. We denote them by $a_{i_1}, \dots, a_{i_k}$, ordered according to their appearance in $a(\underline{j})$; in other words, $a(\underline{j})$ is of the form

$$a(\underline{j}) = \dots a_{i_1} \dots a_{i_2} \dots \dots a_{i_k}.$$

2. Move the $a_{i_1}, \dots, a_{i_k}$ to the far right, without changing their internal order,

$$a(\underline{j}) = a(\underline{j}') \cdot (a_{i_1} a_{i_2} \dots a_{i_k}) = a(\underline{j}') \cdot a(\underline{j}^{(0)})$$

for $\underline{j}^{(0)} = (i_1, \dots, i_k)$ and some sequence $\underline{j}' = (\underline{j}$ with $i_1, \dots, i_k$ removed). This is possible because

   (a) by assumption, there is no a_{i-1} to the right of an a_i in this list;
   (b) if for some i, a_{i+1} occurs to the right of some a_i, then either $a_i \dots a_{i+1} \dots a_i$ would occur as a subword without a_{i-1} in between, hence $a(\underline{j}) = 0$, or else a_{i+1} does not have a_i to its right, so it is one of the $a_{i_1}, \dots, a_{i_k}$ itself, and will be moved to the far right of $a(\underline{j})$, too;
   (c) a_i commutes with all a_ℓ for $\ell \neq i-1, i+1$.

3. Repeat for $a(\underline{j}')$ until we get

$$a(\underline{j}) = a(\underline{j}^{(m)}) \cdot a(\underline{j}^{(m-1)}) \cdot \dots \cdot a(\underline{j}^{(1)}) \cdot a(\underline{j}^{(0)})$$

for sequences $\underline{j}^{(m)}, \dots, \underline{j}^{(1)}$ obtained successively the same way as described above. Notice:

   - Inside a sequence $\underline{j}^{(n)}$, every index occurs at most once. If two consecutive indices occur within $\underline{j}^{(n)}$, they are increasingly ordered using the order $\overset{i_k}{\prec}$ from Definition 8.2.
   - For two consecutive sequences $\underline{j}^{(n+1)}, \underline{j}^{(n)}$ and for every index $i_r^{(n+1)}$ occurring in $\underline{j}^{(n+1)}$, we can find some index $i_s^{(n)}$ in $\underline{j}^{(n)}$ such that $i_r^{(n+1)} = i_s^{(n)} + 1$.
   - From that property, it also follows that the length of $\underline{j}^{(n+1)}$ is less or equal than the length of $\underline{j}^{(n)}$.

4. Reorder the factors $a(\underline{j}^{(m)}), \dots, a(\underline{j}^{(1)}), a(\underline{j}^{(0)})$ internally:

   (a) Start with $a(\underline{j}^{(0)})$. There is some $0 \leq \hat{\imath} \leq N-1$ which does not occur in $\underline{j}^{(0)}$, but $\hat{\imath}-1$ occurs. For example, this is satisfied by $\hat{\imath} = i_k + 1$, as i_k occurs in $\underline{j}^{(0)}$ and is to the right of every other factor of $a(\underline{j})$. Choose the largest such $\hat{\imath}$ (with respect to the usual order). Then we can move $\hat{\imath}-1$ to the very right of the sequence $\underline{j}^{(0)}$, because $\hat{\imath}$ is not present, and $\hat{\imath}-2$ may only occur to the left of $\hat{\imath}-1$ due to the construction of $\underline{j}^{(0)}$. We proceed in the same way with those indices $\hat{\imath}-2, \hat{\imath}-3, \dots, \hat{\imath}-(N-1)$ that appear in $\underline{j}^{(0)}$. The result is a reordering of the sequence $\underline{j}^{(0)}$ so that it is increasing from left to right with respect to $\overset{\hat{\imath}}{\prec}$.
   (b) Repeat with all other factors $a(\underline{j}^{(1)}), a(\underline{j}^{(2)}), \dots, a(\underline{j}^{(m)})$ taking as the initial right-hand index of the sequence $\hat{\imath}, \hat{\imath}+1, \dots, \hat{\imath}+m-1$ respectively, and reordering within each $a(\underline{j}^{(n)})$ so that the indices are increasing from left to right with respect to $\overset{\hat{\imath}+n}{\prec}$. Throughout, the index $\hat{\imath}$ is the one from step (4a).

Example 8.3 As an example for $\mathrm{n}\widehat{\mathrm{TL}}_7$, suppose $a(\underline{j}) = a(6\,4\,2\,1\,3\,5\,4\,2\,0\,6\,1\,3\,2\,5)$. (We omit the commas to simplify the notation.)

Find all a_i without a_{i-1} to their right:	$a(6\,4\,2\,1\,3\,5\,4\,2\,0\,6\,\underline{\underline{1}}\,3\,\underline{\underline{2}}\,\underline{\underline{5}})$
Move them to the far right, and do not change their internal order:	$a(6\,4\,2\,1\,3\,5\,4\,2\,0\,6\,3) \cdot a(1\,2\,5)$
Repeat:	$a(6\,4\,2\,3\,5\,4\,1\,\underline{\underline{2}}\,0\,\underline{\underline{6}}\,\underline{\underline{3}}) \cdot a(1\,2\,5)$
	$a(6\,4\,2\,3\,5\,4\,1\,0) \cdot a(2\,6\,3) \cdot a(1\,2\,5)$ $a(6\,4\,2\,\underline{\underline{3}}\,5\,\underline{\underline{4}}\,1\,\underline{\underline{0}}) \cdot a(2\,6\,3) \cdot a(1\,2\,5)$
	$a(6\,4\,2\,5\,1) \cdot a(3\,4\,0) \cdot a(2\,6\,3) \cdot a(1\,2\,5)$ $a(6\,\underline{4}\,2\,\underline{\underline{5}}\,\underline{\underline{1}}) \cdot a(3\,4\,0) \cdot a(2\,6\,3) \cdot a(1\,2\,5)$
	$a(6\,2) \cdot a(4\,5\,1) \cdot a(3\,4\,0) \cdot a(2\,6\,3) \cdot a(1\,2\,5)$
With the right-hand indices of the $a(\underline{j}^{(n)})$, $n \geq 0$, arranged according to $\hat{\imath}+m-1 \overset{\hat{\imath}}{\succ} \ldots \overset{\hat{\imath}}{\succ} \hat{\imath}+1 \overset{\hat{\imath}}{\succ} \hat{\imath}=6$ from left to right, reorder the factors in each $a(\underline{j}^{(n)})$ increasingly with respect to $\overset{\hat{\imath}+n}{\prec}$ from left to right:	$a(6\,2) \cdot a(4\,5\,1) \cdot a(3\,4\,0) \cdot a(2\,3\,6) \cdot a(1\,2\,5)$

As a shorthand notation, in the following we often identify the index sequence $\underline{j}$ with $a(\underline{j})$ (and manipulate $\underline{j}$ according to the same relations as $a(\underline{j})$) as demonstrated in the following example.

Example 8.4 Let $N = 6$.

$$\begin{aligned}(5\,1\,2\,3\,0\,4\,1\,5\,0\,2\,3\,1\,4\,5\,0\,2\,3\,1\,4\,2) &= (1)(5\,0\,2)(3\,4\,5\,1)(2\,3\,4\,0)(1\,2\,3\,5)(0\,1\,2\,4)\\ &= (1 \quad 5\,0\,2 \quad 3\,4\,5\,1 \quad 2\,3\,4\,0 \quad 1\,2\,3\,5 \quad 0\,1\,2\,4).\end{aligned}$$

Lemma 8.5 *Let $a(\underline{j})$ be a nonzero monomial in $n\widehat{TL}_N$, where we use as always the notation from Section 2. Let $a(\underline{j}^{(m)}), a(\underline{j}^{(m-1)}), \ldots, a(\underline{j}^{(1)}), a(\underline{j}^{(0)})$ be the monomials constructed by the algorithm above.*

1. *The equality $a(\underline{j}) = a(\underline{j}^{(m)})a(\underline{j}^{(m-1)}) \cdots a(\underline{j}^{(1)})a(\underline{j}^{(0)})$ holds in $n\widehat{TL}_N$.*
2. *Given any two representatives $a(\underline{j})$, $a(\underline{j}^{\#})$ of the same element in $n\widehat{TL}_N$, the above algorithm creates the same representative $a(\underline{j}^{(m)})a(\underline{j}^{(m-1)}) \cdots a(\underline{j}^{(1)})a(\underline{j}^{(0)})$ for both $a(\underline{j})$ and $a(\underline{j}^{\#})$.*

Proof 1. The algorithm never interchanges the order of two factors a_i, $a_{i\pm1}$ with consecutive indices within $a(\underline{j})$. Hence, the reordering of the factors of $a(\underline{j})$ uses only the commutativity relation $a_i a_j = a_j a_i$ for $i - j \neq \pm1 \bmod N$ of $\mathrm{n}\widehat{\mathrm{TL}}_N$.

2. Two monomials $a(\underline{j})$, $a(\underline{j}^{\#})$ in $\mathrm{n}\widehat{\mathrm{TL}}_N$ are equal if and only if they only differ by applications of commutativity relations $a_i a_j = a_j a_i$ for $i - j \neq \pm1 \bmod N$, hence, if and only if they contain the same number of factors a_i for each i and the relative position of each a_i and $a_{i\pm1}$ is the same. Since the outcome of the algorithm depends only on the relative positions of consecutive indices, the resulting decomposition $a(\underline{j}^{(m)})a(\underline{j}^{(m-1)}) \cdots a(\underline{j}^{(1)})a(\underline{j}^{(0)})$ is the same.

□

We have shown the following. In stating this result and subsequently, whenever we refer to monomials in normal form, we assume the monomial is nonzero and nonconstant, in particular the sequence $\underline{j}$ is nonempty.

Theorem 8.6 *Assume $N \geq 3$.*

1. *The algorithm in Step 1 above provides a normal form for nonzero monomials $a(\underline{j})$ in the generators a_i of $n\widehat{TL}_N$, or equivalently for nonzero fully commutative monomials in $\widehat{\mathrm{TL}}_N$, so that*

   $$a(\underline{j}) = (a^{(m)}_{i_1} \dots a^{(m)}_{i_k}) \dots (a^{(n+1)}_{i_1} \dots a^{(n+1)}_{i_k})(a^{(n)}_{i_1} \dots a^{(n)}_{i_k}) \dots (a^{(1)}_{i_1} \dots a^{(1)}_{i_k})(a_{i_1} \dots a_{i_k}),$$

   where $a^{(n)}_{i_\ell} \in \{1, a_0, a_1, \dots, a_{N-1}\}$ for all $1 \leq n \leq m,\ 1 \leq \ell \leq k$, and

   $$a^{(n+1)}_{i_\ell} \in \begin{cases} \{1\} & \text{if } a^{(n)}_{i_\ell} = 1, \\ \{1, a_{j+1}\} & \text{if } a^{(n)}_{i_\ell} = a_j. \end{cases}$$

   The factors $a_{i_1}, \dots, a_{i_k}$ are determined by the property that the generator $a_{i_\ell - 1}$ does not appear to the right of a_{i_ℓ} in the original presentation of the monomial. The internal ordering of the factors is increasing with respect to the relation $\overset{\hat{\imath}}{\succ}$, as in Step (4a) of the normal form algorithm, where $\hat{\imath}$ is the largest value in $\{0, 1, \dots, N-1\}$ such that $\hat{\imath} - 1 \notin \{i_1, \dots, i_k\}$, but $\hat{\imath} \in \{i_1, \dots, i_k\}$.
2. *The set $\{a(\underline{j})$ in normal form$\} \cup \{1\}$ is a $\Bbbk$-basis of $n\widehat{TL}_N$.*

8.3 Step 2: Labelling of basis elements

Definition 8.7 Given $a(\underline{j}) = a(\underline{j}^{(m)})a(\underline{j}^{(m-1)}) \cdots a(\underline{j}^{(1)})a(\underline{j}^{(0)})$ in normal form, we call $\underline{j}^{(\ell)}$ the ℓth *block* of $\underline{j}$, and a string of indices of maximal length of the form $i_s \in \underline{j}^{(0)},\ i_s+1 \in \underline{j}^{(1)},\ i_s+2 \in \underline{j}^{(2)}, \dots$ (modulo N) the sth *strand* of $\underline{j}$. We use the notation $[\dots, i_s+1, i_s]$ for the strands.

Example 8.8 Let $N = 6$, and consider Example 8.4 once again, where

$$\underline{j} = (1 \quad 5\,0\,2 \quad 3\,4\,5\,1 \quad 2\,3\,4\,0 \quad 1\,2\,3\,5 \quad 0\,1\,2\,4).$$

The blocks are $\underline{j}^{(0)} = (0124)$, $\underline{j}^{(1)} = (1235)$, $\underline{j}^{(2)} = (2340)$, $\underline{j}^{(3)} = (3451)$, $\underline{j}^{(4)} = (502)$, and $\underline{j}^{(5)} = (1)$. The strands are [3210], [54321], [105432] and [21054]. In particular, strands (and blocks) can have different lengths, but the longest strand has length $m = 6$.

Each monomial $a(\underline{j}) \in n\widehat{\mathrm{TL}}_N$ determines two sets $\mathbf{I}^{\mathrm{in}}_{\underline{j}}, \mathbf{I}^{\mathrm{out}}_{\underline{j}}$ and an integer $\ell_{\underline{j}} \in \mathbb{Z}_{\geq 0}$ as follows:

$$\begin{aligned} \mathbf{I}^{\mathrm{in}}_{\underline{j}} &= \{i \in \{0, 1, \dots, N-1\} \mid \text{no } i-1 \text{ to the right of } i \text{ in } \underline{j}\} \\ \mathbf{I}^{\mathrm{out}}_{\underline{j}} &= \{i \in \{0, 1, \dots, N-1\} \mid \text{no } i+1 \text{ to the left of } i \text{ in } \underline{j}\} \\ \ell_{\underline{j}} &= \text{the number of zeros in } \underline{j}. \end{aligned}$$

These are well defined because, as in the proof of Lemma 8.5, any element of $n\widehat{\mathrm{TL}}_N$ is uniquely determined by the number of factors a_i and the relative position of each a_i and $a_{i\pm 1}$, for all i. The set $\mathbf{I}^{\mathrm{in}}_{\underline{j}}$ equals the underlying set of $\underline{j}^{(0)}$ in the normal form from the algorithm above. All strands of $\underline{j}$ begin with an element in $\mathbf{I}^{\text{in}}_{\underline{j}}$ and end with an element from $\mathbf{I}^{\text{out}}_{\underline{j}}$.

The goal of this subsection is to show

Proposition 8.9 *The mapping*

$$\begin{aligned} \psi : \{a(\underline{j}) \in n\widehat{TL}_N \textit{ in normal form}\} &\to \mathcal{P}_N \times \mathcal{P}_N \times \mathbb{Z}_{\geq 0} \\ a(\underline{j}) &\mapsto (\mathbf{I}^{\text{in}}_{\underline{j}}, \mathbf{I}^{\text{out}}_{\underline{j}}, \ell_{\underline{j}}), \end{aligned} \tag{4}$$

is injective, where $\mathcal{P}_N$ *is the power set of* $\{0, 1, \ldots, N-1\}$.

Remark 8.10 The map ψ is defined so that in the graphical description of the representation V of $\mathrm{n}\widehat{\mathrm{TL}}_N$, the set $\mathbf{I}^{\text{in}}_{\underline{j}}$ equals the set of positions where $a(\underline{j})$ expects a particle to be. The set $\mathbf{I}^{\text{out}}_{\underline{j}}$ equals the set of positions where $a(\underline{j})$ moves the particles from $\mathbf{I}^{\text{in}}_{\underline{j}}$, but each one is translated by 1, that is,

$$a(\underline{j}) \text{ applied to a particle at } i \in \mathbf{I}^{\text{in}}_{\underline{j}} \text{ gives a particle at } j+1 \text{ for some } j \in \mathbf{I}^{\text{out}}_{\underline{j}}.$$

The map ψ is far from being surjective. An obvious constraint is that $|\mathbf{I}^{\text{in}}_{\underline{j}}| = |\mathbf{I}^{\text{out}}_{\underline{j}}|$, and furthermore, for some pairs $(\mathbf{I}^{\text{in}}_{\underline{j}}, \mathbf{I}^{\text{out}}_{\underline{j}})$, one can only obtain sufficiently large values $\ell_{\underline{j}}$.

To ease the presentation, we start by proving injectivity of the restriction ψ_0 of ψ to those monomials $a(\underline{j})$ in normal form whose first element i_1 of $\underline{j}^{(0)}$ is 0. The proof itself will amount to counting indices.

Proposition 8.11 *The map*

$$\begin{aligned} \psi_0 : \{a(\underline{j}) \in n\widehat{TL}_N \textit{ in normal form, with } i_1 = 0\} &\to \mathcal{P}_N \times \mathcal{P}_N \times \mathbb{Z}_{\geq 0}, \\ a(\underline{j}) &\mapsto (\mathbf{I}^{\text{in}}_{\underline{j}}, \mathbf{I}^{\text{out}}_{\underline{j}}, \ell_{\underline{j}}) \end{aligned}$$

is injective.

Before beginning the proof of this result, we note that for monomials $a(\underline{j})$ with $i_1 = 0$, the inequality $i_k < N-1$ must hold in $\mathbf{I}^{\text{in}}_{\underline{j}}$, since $i_1 = 0$ implies that $i_1 - 1 = N-1$ is not an element of $\mathbf{I}^{\text{in}}_{\underline{j}}$. Consequently, the ordering of the indices in $\mathbf{I}^{\text{in}}_{\underline{j}}$ agrees with the natural ordering of $\mathbb{Z}$, so we can regard $(\mathbf{I}^{\text{in}}_{\underline{j}}, <)$ as a subset of $(\mathbb{Z}, <)$ and replace the modular index sequence $\underline{j}$ by an integral index sequence $\underline{j}^{\mathbb{Z}}$ such that $\underline{j}^{\mathbb{Z}} (\bmod N) = \underline{j}$ as follows:

Definition 8.12 Assume $\underline{j} = \underline{j}^{(m)} \cdot \cdots \cdot \underline{j}^{(1)} \cdot \underline{j}^{(0)}$ is a normal form sequence with $\underline{j}^{(0)} = \{0 = i_1 < \cdots < i_k < N-1\}$ and $\underline{j}^{(n)} = (i_{h_1} + n, \ldots, i_{h_{k(n)}} + n) \subseteq (i_1 + n, \ldots, i_k + n)$, where indices in $\underline{j}^{(n)}$ are modulo N and $1 \leq k(n) \leq k$ for all $1 \leq n \leq m$. The *integral normal form sequence for* $\underline{j}$ is

$$\underline{j}^{\mathbb{Z}} = (\underline{j}^{(m)})^{\mathbb{Z}} \cdot \cdots \cdot (\underline{j}^{(1)})^{\mathbb{Z}} \cdot \underline{j}^{(0)} \quad \text{where} \quad (\underline{j}^{(n)})^{\mathbb{Z}} := (i_{h_1} + n, \ldots, i_{h_{k(n)}} + n) \in \mathbb{Z}^{k(n)}$$

for $n = 1, \ldots, m$.

Example 8.13 We continue Example 8.4 with $N = 6$.

$$\begin{aligned} \text{If } \underline{j} &= (1 \quad 5\,0\,2 \quad 3\,4\,5\,1 \quad 2\,3\,4\,0 \quad 1\,2\,3\,5 \quad 0\,1\,2\,4), \\ \text{then } \underline{j}^{\mathbb{Z}} &= (7 \quad 5\,6\,8 \quad 3\,4\,5\,7 \quad 2\,3\,4\,6 \quad 1\,2\,3\,5 \quad 0\,1\,2\,4). \end{aligned}$$

Our proof of Proposition 8.11 will hinge upon the following technical lemma.

Lemma 8.14 *Let $\underline{j}^{\mathbb{Z}}$ be the integral normal form sequence for $\underline{j}$ and let $[i_s, \dots, i_s + n_s]$ for $s = 1, \dots, k$ be the strands of $\underline{j}^{\mathbb{Z}}$. Assume $i_1 = 0$. Then*

(a) $n_1 = i_1 + n_1 < i_2 + n_2 < \cdots < i_k + n_k$;
(b) $i_k + n_k < i_1 + n_1 + N = n_1 + N$.

We postpone the proof of this result and proceed directly to proving the proposition.

Proof (Proposition 8.11) Since the sequence $\underline{j}$ will be fixed throughout the proof, we will drop the subscript $\underline{j}$ on $\mathbf{I}^{\text{in}}_{\underline{j}}\mathbf{I}^{\text{out}}_{\underline{j}}$, $\ell_{\underline{j}}$. To show the injectivity of ψ_0, we consider the factorization $\psi_0 = \gamma \circ \beta \circ \alpha$ given by

$$\psi_0 : a(\underline{j}) \xmapsto{\alpha} a(\underline{j}^{\mathbb{Z}}) \xmapsto{\beta} ((\mathbf{I}^{\text{in}})^{\mathbb{Z}}, (\mathbf{I}^{\text{out}})^{\mathbb{Z}}) \xmapsto{\gamma} (\mathbf{I}^{\text{in}}, \mathbf{I}^{\text{out}}, \ell),$$

where $(\mathbf{I}^{\text{in}})^{\mathbb{Z}} = \mathbf{I}^{\text{in}}$ and $(\mathbf{I}^{\text{out}})^{\mathbb{Z}} = \{i \in \underline{j}^{\mathbb{Z}} \mid \text{no } i+1 \text{ to the left of } i\}$ similar to the definition of $\mathbf{I}^{\text{out}}$. The map α replaces indices in $\mathbb{Z}/N\mathbb{Z}$ by indices in $\mathbb{Z}$ as in Definition 8.12 above. The map β is given by reading off $(\mathbf{I}^{\text{out}})^{\mathbb{Z}}$ and $(\mathbf{I}^{\text{in}})^{\mathbb{Z}}$ from $\underline{j}^{\mathbb{Z}}$. The map γ sends the pair $((\mathbf{I}^{\text{in}})^{\mathbb{Z}}, (\mathbf{I}^{\text{out}})^{\mathbb{Z}})$ to a triple consisting of the respective images $\mathbf{I}^{\text{in}}$, $\mathbf{I}^{\text{out}}$ modulo N of the pair and the integer $\ell = 1 + \sum \ell_r$ where $\ell_r = \lfloor \frac{j_r}{N} \rfloor$ for each $j_r \in (\mathbf{I}^{\text{out}})^{\mathbb{Z}}$. The summand 1 corresponds to $0 = i_1$; all other occurrences of 0 are counted by $\sum \ell_r$. Now we check injectivity.

The map α is clearly injective since $\underline{j}^{\mathbb{Z}} \mapsto \underline{j}^{\mathbb{Z}} (\bmod N)$ is a left inverse map.

To see that β is injective, we need to know that $\underline{j}^{\mathbb{Z}}$ can be uniquely reconstructed from $((\mathbf{I}^{\text{in}})^{\mathbb{Z}}, (\mathbf{I}^{\text{out}})^{\mathbb{Z}})$. Observe that $\underline{j}^{\mathbb{Z}}$ is determined by knowing all the 'strands' $i_s, i_s + 1, i_s + 2, \dots, i_s + n_s$ for $1 \le s \le k$, hence by assigning an element $i_s + n_s \in (\mathbf{I}^{\text{out}})^{\mathbb{Z}}$ to each $i_s \in (\mathbf{I}^{\text{in}})^{\mathbb{Z}}$. But it follows from Lemma 8.14 (a) that $i_1 + n_1$ must be the smallest element of $(\mathbf{I}^{\text{out}})^{\mathbb{Z}}$, $i_2 + n_2$ the second smallest, etc., so that the element $i_s + n_s$ is assigned to the sth element in $\mathbf{I}^{\text{in}}$, that is, to i_s.

Now to see that γ is injective, we need to recover $((\mathbf{I}^{\text{in}})^{\mathbb{Z}}, (\mathbf{I}^{\text{out}})^{\mathbb{Z}})$ in a unique way from $(\mathbf{I}^{\text{in}}, \mathbf{I}^{\text{out}}, \ell)$. Write $\mathbf{I}^{\text{in}} = \{0 = i_1 < \cdots < i_k < N-1\}$, and set $(\mathbf{I}^{\text{in}})^{\mathbb{Z}} := \mathbf{I}^{\text{in}}$. By Lemma 8.14 (a), we know that $(\mathbf{I}^{\text{out}})^{\mathbb{Z}}$ is of the form $(i_1 + n_1 < \cdots < i_k + n_k)$, and since the elements of $\mathbf{I}^{\text{out}}$ have to be equal to the elements of $(\mathbf{I}^{\text{out}})^{\mathbb{Z}}$ modulo N, we can write $i_r + n_r = N\ell_r + d_r$ for $\ell_r = \lfloor \frac{i_r + n_r}{N} \rfloor$ and some $d_r \in \mathbf{I}^{\text{out}}$. Comparing ℓ_r and ℓ_s for $r < s$, we have

$$N\ell_r \le N\ell_r + d_r = i_r + n_r < i_s + n_s = N\ell_s + d_s \le N(\ell_s + 1).$$

So $\ell_r < \ell_s + 1$, i.e. $\ell_r \le \ell_s$. Similarly, we obtain from (b) of Lemma 8.14 that $\ell_k \le \ell_1 + 1$. As a result,

$$N\ell_k \le N\,\ell_k + d_k = i_k + n_k < i_1 + n_1 + N = N(\ell_1 + 1) + d_1 \le N(\ell_1 + 2),$$

i.e. $\ell_k < \ell_1 + 2$. Together we have $\ell_1 = \cdots = \ell_s < \ell_{s+1} = \cdots = \ell_1 + 1$ for some $1 < s \le k$ (where we treat the case $s = k$ by $\ell_1 = \cdots = \ell_k$). Set $\tilde{\ell} := \ell_1$. Then

$$\begin{aligned} i_r + n_r &= N\,\tilde{\ell} + d_r && \text{for } 1 \le r \le s, \\ i_r + n_r &= N(\tilde{\ell} + 1) + d_s && \text{for } s + 1 \le r \le k. \end{aligned}$$

As a first consequence,

$$\ell = 1 + \sum_r \ell_r = 1 + k\tilde{\ell} + (k - s),$$

which determines $\tilde{\ell} = \lfloor \frac{\ell - 1}{k} \rfloor$, and hence all ℓ_r, as well as the index s. Using Lemma 8.14, we determine that

$$i_{s+1} + n_{s+1} < \cdots < i_k + n_k < i_1 + n_1 + N < \cdots < i_s + n_s + N,$$

and so

$$N(\tilde{\ell} + 1) + d_{s+1} < \cdots < N(\tilde{\ell} + 1) + d_k < N(\tilde{\ell} + 1) + d_1 < \cdots < N(\tilde{\ell} + 1) + d_s.$$

Therefore, $d_{s+1} < \cdots < d_k < d_1 < \cdots < d_s$, which fixes the choice of d_r for all r. We conclude that given $(\mathbf{I}^{\text{in}}, \mathbf{I}^{\text{out}}, \ell)$, we can reconstruct $(\mathbf{I}^{\text{out}})^{\mathbb{Z}}$ by setting $i_r + n_r := N\,\ell_r + d_r$. This completes the proof of Proposition 8.11. □

Proof (Lemma 8.14) (a) Let $\underline{j}^{\mathbb{Z}}$ be a nonempty integral normal form sequence with $0 = i_1 < \cdots < i_k \leq N - 1$ and strands $[i_r, \ldots, i_r + n_r]$ for $1 \leq r \leq k$. Assume that there is some index $1 \leq t \leq k - 1$ such that $i_t + n_t \geq i_{t+1} + n_{t+1}$. Since $i_t < i_{t+1}$, we have $n_t > n_{t+1}$. So

$$\underline{j}^{\mathbb{Z}} = \ldots \underbrace{(\ldots\ i_t + n_t\ \ldots)}_{\text{the } n_t\text{th bracket}} \ldots \underbrace{(\ldots\ i_t + n_{t+1} \quad i_{t+1} + n_{t+1}\ \ldots)}_{\text{the } n_{t+1}\text{th bracket}} \ldots .$$

From $i_t + n_{t+1} < i_{t+1} + n_{t+1} \leq i_t + n_t$ it follows that there is some integer $n_{t+1} < p \leq n_t$ such that $i_{t+1} + n_{t+1} = i_t + p$ appears in the strand $[i_t, \ldots, i_t + n_t]$, i.e.

$$\underline{j}^{\mathbb{Z}} = \ldots \underbrace{(\ldots\ i_t + n_t\ \ldots)}_{\text{the } n_t\text{th bracket}} \ldots \underbrace{(\ldots\ i_t + p\ \ldots)}_{\text{the } p\text{th bracket}} \ldots \underbrace{(\ldots\ i_t + n_{t+1} \quad i_{t+1} + n_{t+1}\ \ldots)}_{\text{the } n_{t+1}\text{th bracket}} \ldots$$

with $i_t + p = i_{t+1} + n_{t+1}$. But by the definition of the strands, there is no $i_{t+1} + n_{t+1} + 1$ appearing to the left of $i_{t+1} + n_{t+1}$. Due to Lemma 8.1, we know that (even modulo N) there is no repetition of $i_{t+1} + n_{t+1}$ to the left. Thus $i_t + p = i_{t+1} + n_{t+1}$ is not possible, and we obtain $i_1 + n_1 < i_2 + n_2 < \cdots < i_k + n_k$.

For (b) of Lemma 8.14, assume $i_k + n_k \geq i_1 + n_1 + N$. It is true generally that $N > i_k$, so we get $i_k + n_k \geq i_1 + n_1 + N > i_k + n_1$. Hence $i_1 + n_1 + N = i_k + b$ for some $n_1 < b \leq n_k$, i.e. $i_1 + n_1 + N$ appears in the strand $[i_k, \ldots, i_k + n_k]$ and we have

$$\underline{j}^{\mathbb{Z}} = \ldots \underbrace{(\ldots\ i_k + n_k)}_{\text{the } n_k\text{th bracket}} \ldots \underbrace{(\ldots\ i_k + b\ \ldots)}_{\text{the } b\text{th bracket}} \ldots \underbrace{(i_1 + n_1\ \ldots\ i_k + n_1)}_{\text{the } n_1\text{th bracket}} \ldots .$$

Here it may be that the n_kth bracket and the bth bracket coincide, but in any case, we find that $i_k + b = i_1 + n_1 + N = i_1 + n_1 \bmod N$, and so $i_k + b$ appears to the left of $i_1 + n_1$. By the definition of the strands, there is no $i_1 + n_1 + 1$ to the left of $i_1 + n_1$, and from Lemma 8.1 we deduce that in $\underline{j} = \underline{j}^{\mathbb{Z}} \bmod N$ there is no $i_1 + n_1 \bmod N$ to the left of $i_1 + n_1$ allowed, which leads to a contradiction. Hence $i_k + n_k < i_1 + n_1 + N$ must hold. □

Having established that ψ is injective when restricted to sequences with $i_1 = 0$, we now show the injectivity of ψ in general.

Proof (Proposition 8.9) We have the following disjoint decompositions according to the smallest value i_1 in $\underline{j}^{(0)}$ for $\underline{j}$:

$$\begin{aligned}
\{a(\underline{j}) \text{ in normal form}\} &= \coprod_i \{a(\underline{j}) \text{ in normal form, with } i_1 = i\} \\
\{(\mathbf{I}^{\text{in}}_{\underline{j}}, \mathbf{I}^{\text{out}}_{\underline{j}}, \ell_{\underline{j}})\} &= \coprod_i \{(\mathbf{I}^{\text{in}}_{\underline{j}}, \mathbf{I}^{\text{out}}_{\underline{j}}, \ell_{\underline{j}}) \mid i_1 = i \in \mathbf{I}^{\text{in}}_{\underline{j}}\} \\
\psi &= \coprod_i \Big(\psi_i : \{a(\underline{j}) \text{ in normal form, with } i_1 = i\} \\
&\qquad \to \{(\mathbf{I}^{\text{in}}_{\underline{j}}, \mathbf{I}^{\text{out}}_{\underline{j}}, \ell_{\underline{j}}) \mid i_1 = i \in \mathbf{I}^{\text{in}}_{\underline{j}}\}\Big).
\end{aligned}$$

By Proposition 8.11, the map $\psi_0 : a(\underline{j}) \mapsto (\mathbf{I}^{\text{in}}_{\underline{j}}, \mathbf{I}^{\text{out}}_{\underline{j}}, \ell_{\underline{j}})$ restricted to those $a(\underline{j})$ with $i_1 = 0$ is injective. We argue next that by an index shift this result is true for all other ψ_i.

Now it follows from Proposition 8.11 that the map

$$\widehat{\psi}_0 : \{a(\underline{j}) \in \mathrm{n}\widehat{\mathrm{TL}}_N \text{ in normal form, with } i_1 = 0\} \to \{(\mathbf{I}^{\text{in}}_{\underline{j}}, \mathbf{I}^{\text{out}}_{\underline{j}}, \widehat{\ell_{\underline{j}}}) \mid i_1 = 0 \in \mathbf{I}_{\text{in}}\}$$

is injective, where $\widehat{\ell_{\underline{j}}}$ counts the occurences of $N - i$ in $\underline{j}$. Recall that

$\ell_{\underline{j}} = \sum_r \ell_r + 1$ and ℓ_r is the number of zeros in the rth strand $[i_r, \dots, i_r + n_r]$ of $\underline{j}$ mod N.

Now observe that we can obtain $\ell_{\underline{j}}$ from $\widehat{\ell_{\underline{j}}}$ as

$$\ell_{\underline{j}} = \widehat{\ell_{\underline{j}}} - \left|\{d_r \in \mathbf{I}^{\text{out}}_{\underline{j}} \mid d_r \geq N - i\}\right| + \left|\{i_r \in \mathbf{I}^{\text{in}}_{\underline{j}} \mid i_r > N - i\}\right| + 1,$$

which follows from a computation using $\widehat{\ell_{\underline{j}}} = \sum_r \widehat{\ell_r}$ and

$$\begin{aligned}
\widehat{\ell_r} &= \text{ the number of } N - i \text{ in the } r\text{th strand } [i_r, \dots, i_r + n_r] \bmod N \\
&= \begin{cases} \lfloor \frac{i_r + n_r + i}{N} \rfloor & \text{if } i_r \leq N - i \\ \lfloor \frac{i_r + n_r + i}{N} \rfloor - 1 & \text{if } i_r > N - i \end{cases} \\
&= \begin{cases} \lfloor \frac{N\ell_r + d_r + i}{N} \rfloor & \text{if } i_r \leq N - i \\ \lfloor \frac{N\ell_r + d_r + i}{N} \rfloor - 1 & \text{if } i_r > N - i \end{cases} \\
&= \begin{cases} \ell_r + 1 & \text{if } i_r \leq N - i \text{ and } d_r + i \geq N \\ \ell_r & \text{if } i_r \leq N - i \text{ and } d_r + i < N \\ \ell_r & \text{if } i_r > N - i \text{ and } d_r + i \geq N \\ \ell_r - 1 & \text{if } i_r > N - i \text{ and } d_r + i < N. \end{cases}
\end{aligned}$$

We obtain ψ_i by first shifting the indices of $\underline{j}$ by subtracting i from each index, $\underline{j} - (i, \dots, i)$, then applying $\widehat{\psi}_0$, and finally shifting the indices from $\mathbf{I}^{\text{in}}_{\underline{j}}$ and $\mathbf{I}^{\text{out}}_{\underline{j}}$ by adding i to each. Hence, ψ_i is injective for each i, and ψ is injective because the unions are disjoint. □

8.4 Step 3: Description and linear independence of the matrices

Recall that the standard $\Bbbk$-basis of the representation $\mathsf{V} = \bigoplus_{k=0}^N \left(\Bbbk[q] \otimes \bigwedge^k \Bbbk^N\right)$ is given by

$$\{q^\ell \cdot v_{i_1} \wedge \cdots \wedge v_{i_k} \mid \ell \in \mathbb{Z}_{\geq 0},\ 1 \leq i_1 < \cdots < i_k \leq N\}$$

where $(i_1, \ldots, i_k)$ is identified with the particle configuration having particles in those positions in the graphical description. Now we describe with respect to this basis the matrix representing a nonzero monomial $a(\underline{j}) \in \mathrm{n}\widehat{\mathrm{TL}}_N$ as a $2^N \times 2^N$-matrix with entries in $\Bbbk[q]$. Since V decomposes as a $\mathrm{n}\widehat{\mathrm{TL}}_N$-module into submodules $\Bbbk[q] \otimes \bigwedge^k \Bbbk^N$ for $k = 0, 1, \ldots, N$, the matrix of $a(\underline{j})$ is block diagonal with $N + 1$ blocks $A_0, A_1, \ldots, A_N$, where $A_0 = A_N = (0)$ corresponding to the trivial representation.

$$a(\underline{j}) = \begin{pmatrix} 0 & 0 & & \cdots & 0 \\ 0 & \boxed{A_1} & & & \vdots \\ & & \ddots & & \\ \vdots & & & \boxed{A_{N-1}} & 0 \\ 0 & \cdots & & 0 & 0 \end{pmatrix}$$

The block A_k is a $\binom{N}{k} \times \binom{N}{k}$-matrix, with entries from $\Bbbk[q]$ indexed by all possible particle configurations whose number of particles equal to k.

Now fix a nonzero monomial $a(\underline{j})$ in normal form that is specified by the triple $(\mathbf{I}^{\mathrm{in}}_{\underline{j}}, \mathbf{I}^{\mathrm{out}}_{\underline{j}}, \ell_{\underline{j}})$ defined in Step 2. Let $k = |\mathbf{I}^{\mathrm{in}}_{\underline{j}}|$. All blocks $A_1, \ldots, A_{k-1}$ are zero since $a(\underline{j})$ expects at least k particles. For $r > k$ there might be nonzero blocks A_r. Such nonzero blocks appear unless the particles from $\mathbf{I}^{\mathrm{in}}_{\underline{j}}$ are moved around the whole circle with no position left out, in which case there are no surplus particles allowed. This occurs if $a(\underline{j})$ contains at least every other generator $a_i, a_{i+2}, \ldots$.

More importantly, the block A_k has precisely one nonzero entry, and this is given by

$$(A_k)_{\mathbf{I}^{\mathrm{in}}_{\underline{j}}, \mathbf{I}^{\mathrm{out}}_{\underline{j}}} = \pm q^{\ell_{\underline{j}}}.$$

From this we see first that all matrices representing monomials $a(\underline{j})$ in normal form with $|\mathbf{I}^{\mathrm{in}}_{\underline{j}}| = N - 1$ are $\Bbbk$-linearly independent: They have only one nonzero entry which is equal to $\pm q^{\ell_{\underline{j}}}$ at position $(\mathbf{I}^{\mathrm{in}}_{\underline{j}}, \mathbf{I}^{\mathrm{out}}_{\underline{j}})$. Furthermore, if all matrices representing monomials $a(\underline{j})$ in normal form with $|\mathbf{I}^{\mathrm{in}}_{\underline{j}}| \geq k$ are $\Bbbk$-linearly independent, then also all matrices representing monomials $a(\underline{j})$ in normal form with $|\mathbf{I}^{\mathrm{in}}_{\underline{j}}| \geq k - 1$ are $\Bbbk$-linearly independent. This follows because the additional monomials $a(\underline{j})$ with $|\mathbf{I}^{\mathrm{in}}_{\underline{j}}| = k - 1$ have nonzero entries $(A_{k-1})_{\mathbf{I}^{\mathrm{in}}_{\underline{j}}, \mathbf{I}^{\mathrm{out}}_{\underline{j}}} = \pm q^{\ell_{\underline{j}}}$ in the $(k - 1)$th block which is zero for all $a(\underline{j})$ with $|\mathbf{I}^{\mathrm{in}}_{\underline{j}}| \geq k$. So by induction, all matrices representing monomials $a(\underline{j})$ in normal form are $\Bbbk$-linearly independent. Since all of them have a zero entry in the upper left (and lower right) corner, we may add the identity matrix to the linearly independent set of matrices, and it remains linearly independent. So the representation of $\mathrm{n}\widehat{\mathrm{TL}}_N$ on V is faithful, because according to Theorem 8.6, $\{a(\underline{j}) \text{ in normal form}\} \cup \{1\}$ is a $\Bbbk$-basis of $\mathrm{n}\widehat{\mathrm{TL}}_N$.

Section 8 has given a normal form for each monomial and has provided an alternate proof of the faithfulness of the representation of $\mathrm{n}\widehat{\mathrm{TL}}_N$ by elementary arguments.

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